Colimits of Double Categories

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Double Categories

- A **double category** is an internal category in **Cat**,

$$
\begin{tikzcd}
\mathbf{C}_1 \times_{\mathbf{C}_0} \mathbf{C}_1 \arrow[r, "\circ"] & \mathbf{C}_1 \arrow[r, shift left=2, "s"] \arrow[r, shift right=2, "t"'] & \mathbf{C}_0 \arrow[l, "i^{\circ}" description]
\end{tikzcd} .
$$

- Since $\mathbf{C}_0$ and $\mathbf{C}_1$ are categories, this is really a diagram

$$
\begin{tikzcd}
 & \mathbf{C}_{11} \times_{\mathbf{C}_{10}} \mathbf{C}_{11} \arrow[d, "\bullet"'] & \mathbf{C}_{01} \times_{\mathbf{C}_{00}} \mathbf{C}_{01} \arrow[d, "\bullet"'] \\
\mathbf{C}_{11} \times_{\mathbf{C}_{01}} \mathbf{C}_{11} \arrow[r, "\circ"] & \mathbf{C}_{11} \arrow[r, shift left=2, "s"] \arrow[r, shift right=2, "t"'] \arrow[d, shift right=2, "d_1"'] \arrow[d, shift left=2, "d_0"] & \mathbf{C}_{01} \arrow[l, "i^{\circ}" description] \arrow[d, shift right=2, "d_1"'] \arrow[d, shift left=2, "d_0"] \\
\mathbf{C}_{10} \times_{\mathbf{C}_{00}} \mathbf{C}_{10} \arrow[r, "\circ"] & \mathbf{C}_{10} \arrow[u, "i^{\bullet}" description] \arrow[r, shift left=2, "s"] \arrow[r, shift right=2, "t"'] & \mathbf{C}_{00} \arrow[l, "i^{\circ}" description] \arrow[u, "i^{\bullet}" description]
\end{tikzcd}
$$

Double Categories

In other words, a double category $\mathbb{D}$ has

- **objects** $\mathbf{C}_{00}$,
- **vertical arrows** $\mathbf{C}_{01}$, denoted $d_0(v) \xrightarrow{v} d_1(v)$,
- **horizontal arrows** $\mathbf{C}_{10}$, denoted $s(f) \xrightarrow{f} t(f)$,
- **double cells** $\mathbf{C}_{11}$, denoted

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"', "\bullet" marking] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "v", "\bullet" marking] \\ A' \arrow[r, "f'"'] & B' \end{tikzcd}$$

where $d_0(\alpha) = f$, $d_1(\alpha) = f'$, $s(\alpha) = u$, and $t(\alpha) = v$.

Double Cell Composition

Double cells can be composed

- **Horizontally**

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] \arrow[dr, phantom, "\alpha"] & B \arrow[r, "g"] \arrow[d, "v"'] \arrow[dr, phantom, "\beta"] & C \arrow[d, "w"] \\ A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' \end{tikzcd} \mapsto \begin{tikzcd} A \arrow[r, "gf"] \arrow[d, "u"'] \arrow[dr, phantom, "\beta\circ\alpha"] & C \arrow[d, "w"] \\ A' \arrow[r, "g'f'"'] & C' \end{tikzcd}$$

- **Vertically**

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "v"] \\ A' \arrow[r, "f'"] \arrow[d, "u'"'] \arrow[dr, phantom, "\alpha'"] & B' \arrow[d, "v'"] \\ A'' \arrow[r, "f''"'] & B'' \end{tikzcd} \mapsto \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u'u"'] \arrow[dr, phantom, "\alpha'\bullet\alpha"] & B \arrow[d, "v'v"] \\ A'' \arrow[r, "f''"'] & B'' \end{tikzcd}$$

Double Cell Composition

Both composition operations are required to be associative and together they need to satisfy the middle-four axiom:

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "u"', "\bullet" marking] & B \arrow[r, "g"] \arrow[d, "v"', "\bullet" marking] & C \arrow[d, "w", "\bullet" marking] \\
A' \arrow[r, "f'"] \arrow[d, "u'"', "\bullet" marking] & B \arrow[r, "g'"] \arrow[d, "v'"', "\bullet" marking] & C \arrow[d, "w'", "\bullet" marking] \\
A'' \arrow[r, "f''"'] & B'' \arrow[r, "g''"'] & C''
\arrow[from=1-1, to=2-2, phantom, "\alpha"]
\arrow[from=1-2, to=2-3, phantom, "\beta"]
\arrow[from=2-1, to=3-2, phantom, "\alpha'"]
\arrow[from=2-2, to=3-3, phantom, "\beta'"]
\end{tikzcd}
$$

$$(\beta' \circ \alpha') \bullet (\beta \circ \alpha) = (\beta' \bullet \beta) \circ (\alpha' \bullet \alpha).$$

Examples

1. For any 2-category $\mathcal{C}$, $\mathbb{Q}(\mathcal{C})$ is the double category of quintets in $\mathcal{C}$, with double cells

$$\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d, "u"'] \arrow[dr, phantom, "\alpha"] & \cdot \arrow[d, "v"] \\ \cdot \arrow[r, "g"'] & \cdot \end{tikzcd}$$

for each $\alpha \colon vf \Rightarrow gu$ in $\mathcal{C}$.

2. For any 2-category $\mathcal{C}$, $\mathbb{H}(\mathcal{C})$ is the double category with double cells

$$\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d, "1_A"'] \arrow[dr, phantom, "\alpha"] & \cdot \arrow[d, "1_B"] \\ \cdot \arrow[r, "g"'] & \cdot \end{tikzcd}$$

for each $\alpha \colon f \Rightarrow g$ in $\mathcal{C}$.

3. The double category $\mathbb{V}(\mathcal{C})$ is defined analogously.

Example: Matched Pairs of Groups

- Let Σ, F and G be groups with $\Sigma = FG$ with right action $\triangleleft\colon G \times F \to G$ and left action $\triangleright\colon G \times F \to F$ defined by

$$g \cdot f = (g \triangleright f) \cdot (f \triangleleft g)$$

such that

$$\begin{aligned} g \triangleright (f_2 f_1) &= (g \triangleright f_2) \cdot ((g \triangleleft f_2) \triangleright f_1) \\ (g_2 g_1) \triangleleft f &= (g_2 \triangleleft (g_1 \triangleright f)) \cdot (g_1 \triangleleft f) \end{aligned}$$

- We can model this as a double category

$$\begin{tikzcd} G \times F \arrow[r, shift left, "s=\pi_2"] \arrow[r, shift right, "t=\triangleright"'] \arrow[d, shift right, "d_1=\pi_1"'] \arrow[d, shift left, "d_0=\triangleleft"] & F \arrow[d, shift right] \arrow[d, shift left] \\ G \arrow[r, shift left] \arrow[r, shift right] & \{\bullet\} \end{tikzcd}$$

Example: Matched Pairs of Groups

- Double cells are of the form

$$\begin{tikzcd} \bullet \arrow[r, "g \triangleleft f"] \arrow[d, "f"'] \arrow[dr, phantom, "(g,f)"] & \bullet \arrow[d, "g \triangleright f"] \\ \bullet \arrow[r, "g"'] & \bullet \end{tikzcd}$$

- Note: for each left-hand corner there is precisely one double cell.
- Such double categories are called *vacant*.
- Horizontal composition:

$$\begin{tikzcd} \bullet \arrow[r, "g_1 \triangleleft f"] \arrow[d, "f"'] & \bullet \arrow[r, "g_2 \triangleleft (g_1 \triangleright f)"] \arrow[d, "g_1 \triangleright f"'] & \bullet \arrow[d, "g_2 \triangleright (g_1 \triangleright f))"] \\ \bullet \arrow[r, "g_1"'] & \bullet \arrow[r, "g_2"'] & \bullet \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} \bullet \arrow[r, "(g_2 g_1) \triangleleft f"] \arrow[d, "f"'] & \bullet \arrow[d, "(g_2 g_1) \triangleright f"] \\ \bullet \arrow[r, "g_2 g_1"'] & \bullet \end{tikzcd}$$

  since $(g_2 g_1) \triangleleft f = (g_2 \triangleleft (g_1 \triangleright f)) \cdot (g_1 \triangleleft f)$.
- Vertical composition goes similarly (using the other condition).

Matched Pairs of Groupoids

- This leads us to a straightforward generalization of the notion of matched pair of groups; namely, a *matched pair of groupoids*.
- A matched pair of groupoids is a pair of groupoids

$$d_0, d_1 \colon \mathcal{V} \rightrightarrows \mathcal{P} \quad \text{and} \quad s, t \colon \mathcal{H} \rightrightarrows \mathcal{P}$$

  with the same base (set of objects) with actions

$$\triangleright \colon \mathcal{H} \times_{s,\mathcal{P},d_1} \mathcal{V} \to \mathcal{V} \quad \text{and} \quad \triangleleft \colon \mathcal{H} \times_{s,\mathcal{P},d_1} \mathcal{V} \to \mathcal{H}$$

  such that we can form double cells

$$\begin{tikzcd} P \arrow[r, "h \triangleleft v"] \arrow[d, "v"'] \arrow[dr, phantom, "(v,h)"] & Q \arrow[d, "h \triangleright v"] \\ R \arrow[r, "h"'] & S \end{tikzcd}$$

  and horizontal and vertical composition are well-defined.
- Result: Matched pair of groupoids are vacant double groupoids.

The category **DblCat**

The category **DblCat** of double categories has:

- objects: double categories $\mathbb{C}, \mathbb{D}, \ldots$;
- arrows: double functors $F, G, \ldots$ are internal functors,

$$\begin{tikzcd} \mathbf{C}_1 \times_{\mathbf{C}_0} \mathbf{C}_1 \arrow[r, "\circ"] \arrow[d, "F_1 \times F_1"'] & \mathbf{C}_1 \arrow[r, shift left=2, "s"] \arrow[r, shift right=2, "t"'] \arrow[d, "F_1"'] & \mathbf{C}_0 \arrow[l, "i^\circ" description] \arrow[d, "F_0"] \\ \mathbf{D}_1 \times_{\mathbf{D}_0} \mathbf{D}_1 \arrow[r, "\circ"] & \mathbf{D}_1 \arrow[r, shift left=2, "s"] \arrow[r, shift right=2, "t"'] & \mathbf{D}_0 \arrow[l, "i^\circ" description] \end{tikzcd}$$

  where F_0 and F_1 are functors.
- 2-cells: these come in two flavours: internal and external; or, vertical and horizontal.

Transformations

- Vertical Transformations $\gamma\colon F \Rightarrow G\colon \mathbb{C} \rightrightarrows \mathbb{D}$ given by

$$\begin{tikzcd} FA \arrow[r, "Fh"] \arrow[d, "\gamma_A"'] \arrow[dr, phantom, "\gamma_h"] & FB \arrow[d, "\gamma_B"] \\ GA \arrow[r, "Gh"'] & GB \end{tikzcd} \quad \text{for each } h\colon A \to B \text{ in } \mathbb{C}$$

  functorial in the horizontal direction and natural in the vertical direction.

- Horizontal Transformations $\nu\colon F \Longrightarrow G$ are defined dually, by a family of double cells,

$$\begin{tikzcd} FA \arrow[r, "\nu_A"] \arrow[d, "Fu"'] \arrow[dr, phantom, "\nu_u"] & GA \arrow[d, "Gu"] \\ FA' \arrow[r, "\nu_{A'}"'] & GA' \end{tikzcd}$$

Modifications

Modifications are 3-dimensional cells

$$\begin{tikzcd}
F \arrow[r, Rightarrow, "\mu"] \arrow[d, Rightarrow, "\gamma"', "\bullet" marking] \arrow[dr, phantom, "\Theta"] & G \arrow[d, Rightarrow, "\delta", "\bullet" marking] \\
F' \arrow[r, Rightarrow, "\nu"'] & G'
\end{tikzcd}$$

that are given by a family of double cells, indexed by the objects of the domain double category,

$$\begin{tikzcd}
FA \arrow[r, "\mu_A"] \arrow[d, "\gamma_A"', "\bullet" marking] \arrow[dr, phantom, "\Theta_A"] & GA \arrow[d, "\delta_A", "\bullet" marking] \\
F'A \arrow[r, "\nu_A"'] & G'A.
\end{tikzcd}$$

Interlude

- When we view a group G as a one-object category BG, functors $BG \to BH$ correspond precisely to group homomorphisms $G \to H$.
- Now the natural transformations make the category of groups into a 2-category: natural transformations $B\varphi \Rightarrow B\psi$ correspond to group elements $h \in H$ such that $h\psi h^{-1} = \varphi$.
- This also places groups into a much larger category of groupoids or categories and this means that the notion of colimit of a diagram of groups may change significantly:
    - The colimit of a disconnected diagram is not a group.
    - We may also consider pseudo and lax colimits. (An important application of this is the tom Dieck fundamental groupoid.)

Example

- Double functors between vacant double groupoids correspond to groupoid homomorphisms between matched pairs that preserve the factorization.
- For matched pairs of groups, vertical transformations are determined by an element of the first factor of the codomain that establishes a conjugation between the homomorphisms between the first factors of the matched pairs.
- Analogously, horizontal transformations are determined by conjugation with an element of the second factor of the codomain.
- Modifications correspond to a pair of a horizontal and a vertical transformation.

The category **DblCat** - Properties

- **DblCat** is not a double category.
- **DblCat** is enriched in the category **DblCat** of double categories: each **DblCat**$(\mathbb{C}, \mathbb{D})$ is a double category.
- We can also view **DblCat** as a 2-category: $\mathbf{DblCat}_v$ (resp. $\mathbf{DblCat}_h$) is the 2-category with vertical (resp. horizontal) transformations.
- So lax (co)limits have typically been taken in the 2-category $\mathbf{DblCat}_v$ or $\mathbf{DblCat}_h$ with laxity in one direction.
- We want to introduce a notion of colimit that may have laxity in both directions - so we want to index the diagram by a double category.

Diagrams in **DblCat**

To define a diagram of double categories indexed by a double category $\mathbb{D}$:

- Send objects of $\mathbb{D}$ to double categories;
- Send both horizontal and vertical arrows to double functors;
- For 2-dimensional cells we have to make a choice: we send double cells to *vertical* transformations.

So an indexing double functor is a double functor

$$\mathbb{D} \to \mathbb{Q}(\mathbf{DblCat}_v)$$

We will also refer to indexing double functors as **vertical double functors**

$$\mathbb{D} \xrightarrow{v} \mathbf{DblCat}.$$

Questions

- Have we lost our ability to use horizontal transformations and modifications?
- Have we lost our ability to distinguish between horizontal and vertical arrows in the indexing double category?

No, they will show up in the notion of **doubly lax transformation**.

Intro to Doubly Lax Transformations

- Introduce a **cylinder double category** $\mathsf{Cyl}_v(\mathbf{DblCat})$.
- There are vertical double functors

$$\mathsf{Cyl}_v(\mathbf{DblCat}) \underset{d_1}{\overset{d_0}{\rightrightarrows}} \mathbf{DblCat}$$

- A **doubly lax transformation** $\alpha\colon F \Rightarrow G\colon \mathbb{D} \to \mathbf{DblCat}$ is given by a double functor

$$\alpha\colon \mathbb{D} \to \mathsf{Cyl}_v(\mathbf{DblCat})$$

  such that $d_0\alpha = F$ and $d_1\alpha = G$.

The Double Category of (Vertical) Cylinders

The double category $\mathrm{Cyl}_v(\mathbf{DblCat})$ of **vertical cylinders** is defined by:

- Objects are double functors, denoted by $\downarrow f$.
- Vertical arrows $f \xrightarrow{(u,\mu,v)} \overline{f}$ are given by vertical transformations,

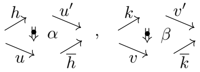
$$
\begin{tikzcd}
{} \arrow[r, "u"] \arrow[d, "f"'] \arrow[dr, phantom, "\Uparrow\!\bullet\,\mu"] & {} \arrow[d, "\overline{f}"] \\
{} \arrow[r, "v"'] & {}
\end{tikzcd}
$$

- Horizontal arrows $f \xrightarrow{(h,\kappa,k)} f'$ are given by horizontal transformations,

$$
\begin{tikzcd}
{} \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, phantom, "\Uparrow\kappa"] & {} \arrow[d, "f'"] \\
{} \arrow[r, "k"'] & {}
\end{tikzcd}
$$

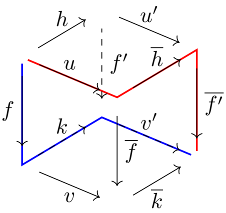
Double Cylinders

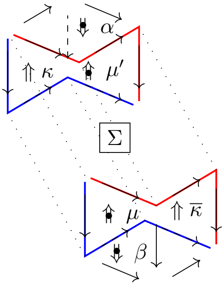
A double cell,
$$\begin{tikzcd} f \arrow[r, "{(h,\kappa,k)}"] \arrow[d, "{(u,\mu,v)}"'] \arrow[dr, phantom, "{(\alpha,\Sigma,\beta)}"] & f' \arrow[d, "{(u',\mu',v')}"] \\ \overline{f} \arrow[r, "{(\overline{h},\overline{\kappa},\overline{k})}"'] & \overline{f'} \end{tikzcd}$$
consists of two vertical 2-cells,



and a modification Σ,

$$\begin{tikzcd} v'kf \arrow[r, Rightarrow, "v'\kappa"] \arrow[d, Rightarrow, "\beta f"] \arrow[ddr, phantom, "\Sigma"] & v'f'h \arrow[d, Rightarrow, "\mu' h"] \\ \overline{k}vf \arrow[d, Rightarrow, "\overline{k}\mu"] & \overline{f'}u'h \arrow[d, Rightarrow, "\overline{f'}\alpha"] \\ \overline{k}\,\overline{f}u \arrow[r, Rightarrow, "\overline{\kappa}u"'] & \overline{f'}\,\overline{h}u \end{tikzcd}$$





Cylinders and Transformations

- There are vertical double functors $d_0, d_1 \colon \mathsf{Cyl}_v(\mathbf{DblCat}) \xrightarrow{v} \mathbf{DblCat}$, sending a cylinder to its top and bottom respectively;
- A **doubly lax transformation** $\theta \colon F \Rightarrow G$ between vertical double functors $F, G \colon \mathbb{D} \xrightarrow{v} \mathbf{DblCat}$ is given by a double functor

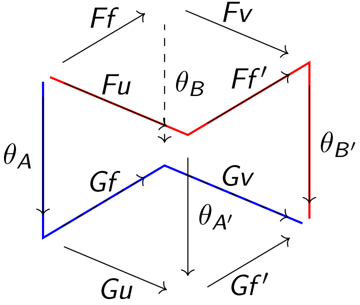
$$\theta \colon \mathbb{D} \to \mathsf{Cyl}_v(\mathbf{DblCat}),$$

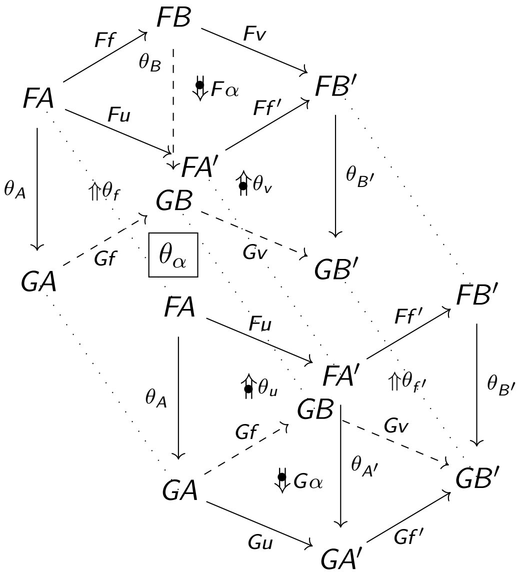
  such that $d_0\theta = F$ and $d_1\theta = G$.

Doubly Lax Transformations $\theta\colon F \Rightarrow G$

For each double cell
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "v"] \\ A' \arrow[r, "f'"'] & B' \end{tikzcd},$$

$$\begin{tikzcd} {\color{blue}GvGf\theta_A} \arrow[r, Rightarrow, "Gv\theta_f"] \arrow[d, Rightarrow, "G\alpha\theta_A"'] \arrow[dr, phantom, "\theta_\alpha"] & Gv\theta_B Ff \arrow[d, Rightarrow, "\theta_v Ff"] \\ Gf' Gu\theta_A \arrow[d, Rightarrow, "Gf'\theta_u"'] & \theta_{B'} FvFf \arrow[d, Rightarrow, "\theta_{B'} F\alpha"] \\ Gf'\theta_{A'} Fu \arrow[r, Rightarrow, "\theta_{f'} Fu"'] & {\color{red}\theta_{B'} Ff' Fu} \end{tikzcd}$$





FB
Ff
Fv
θ_B
Fα
FB′
FA
Fu
Ff′
FA′
θ_A
θ_f
θ_v
θ_B′
GB
GA
Gf
θ_α
Gv
GB′
FA
Fu
FB′
Ff′
FA′
θ_A
θ_u
θ_f′
θ_B′
GB
Gf
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Gα
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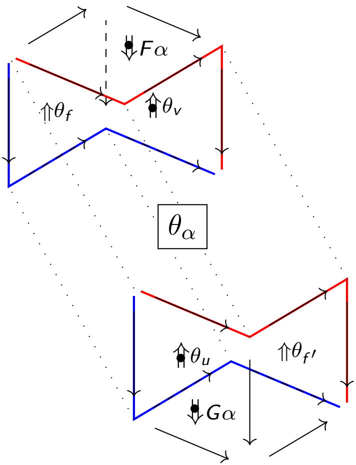
GB′
Gu
Gf′
GA′



Fα
θ_f
θ_v
θ_α
θ_u
θ_f′
Gα

Doubly Lax Transformations

- Let $F, G \colon \mathbb{D} \xrightarrow{v} \mathbf{DblCat}$ be vertical double functors.
- Since doubly lax transformations $F \Rightarrow G$ are represented by double functors,

$$\mathbb{D} \to \mathsf{Cyl}_v(\mathbf{DblCat})$$

they are the objects of a double category

$$\mathbb{H}\mathrm{om}_{d\ell}(F, G),$$

a sub double category of $\mathbf{DblCat}(\mathbb{D}, \mathsf{Cyl}_v(\mathbf{DblCat}))$.

Lax Transformations Between 2-Functors

- By applying $\mathbb{Q}$ to the hom-categories of a 2-category $\mathcal{B}$, we can make it into a **DblCat**-enriched category $\widehat{\mathbf{Q}}(\mathcal{B})$.
- This allows us to view lax transformations between 2-functors as a special case of the new doubly lax transformations.

$$
\begin{tikzcd}
\mathcal{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow \alpha"] & \mathcal{B}
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
\mathbb{Q}\mathcal{A} \arrow[r, bend left, "\mathbb{Q}F", "v" description] \arrow[r, bend right, "\mathbb{Q}G"', "v" description] \arrow[r, phantom, "\Downarrow \alpha"] & \widehat{\mathbf{Q}}(\mathcal{B})
\end{tikzcd}
$$

- By taking a restricted $\mathbb{Q}$ on the codomain, taking only a particular class Ω of 2-cells of $\mathcal{B}$ for the local horizontal arrows, we obtain Ω-transformations.
- By taking a restricted $\mathbb{Q}$ on the domain, we also get Σ-transformations.

Doubly Lax Colimits

- A **doubly lax cocone** for a vertical double functor $F : \mathbb{D} \nrightarrow \mathbf{DblCat}$ with vertex $\mathbb{E} \in \mathbf{DblCat}$ is a doubly lax transformation $F \overset{\theta}{\Rightarrow} \Delta\mathbb{E}$.
- There is a double category,

$$\mathbb{L}\mathrm{C}(F, \mathbb{E}) := \mathbb{H}\mathrm{om}_{d\ell}(F, \Delta\mathbb{E})$$

  of doubly lax cocones with vertex $\mathbb{E}$.
- A doubly lax cocone $F \overset{\lambda}{\Longrightarrow} \Delta\mathbb{L}$ is the **doubly lax colimit** of F if, for every $\mathbb{E} \in \mathbf{DblCat}$,

$$\mathbf{DblCat}(\mathbb{L}, \mathbb{E}) \xrightarrow{\lambda^*} \mathbb{L}\mathrm{C}(F, \mathbb{E})$$

  is an isomorphism of double categories.
- The doubly lax colimit can be obtained by a **double Grothendieck construction**.

The Double Grothendieck Construction: Objects and Arrows

Let $\mathbb{D} \xrightarrow[v]{F} \mathbf{DblCat}$ be a vertical double functor. The **double category of elements**, $\mathbb{G}\mathrm{r}\, F = \int_{\mathbb{D}} F$, is defined by:

- Objects: (C, x) with C in $\mathbb{D}$ and x in FC,
- Vertical arrows:
$$(C, x) \xrightarrow{(u,\rho)} (C', x'),$$
where $C \xrightarrow{u} C'$ in $\mathbb{D}$ and $Fux \xrightarrow{\rho} x'$ in FC'.
- Horizontal arrows:
$$(C, x) \xrightarrow{(f,\varphi)} (D, y),$$
where $C \xrightarrow{f} D$ in $\mathbb{D}$, and $Ffx \xrightarrow{\varphi} y$ in FD.

The Double Grothendieck Construction: Double Cells

- Double cells: $$\begin{tikzcd} (C,x) \arrow[r, "{(f,\varphi)}"] \arrow[d, "{(u,\rho)}"'] \arrow[dr, phantom, "{(\alpha,\Phi)}"] & (D,y) \arrow[d, "{(v,\lambda)}"] \\ (C',x') \arrow[r, "{(f',\varphi')}"'] & (D',y') \end{tikzcd}$$, where $\alpha\colon (u \ {}^{f}_{f'} \ v)$ is a double cell in $\mathbb{D}$ and Φ is a double cell in FD':

$$\begin{tikzcd} FvFfx \arrow[r, "Fv\varphi"] \arrow[d, "(F\alpha)_x"'] & Fvy \arrow[dd, "\lambda"] \\ Ff'Fux \arrow[d, "Ff'\rho"'] \arrow[r, phantom, "\Phi"] & \\ Ff'x' \arrow[r, "\varphi'"'] & y' \end{tikzcd}$$

Factorization

- Any horizontal arrow (f,φ) can be factored as $(A,x) \xrightarrow{(f,1_{Ffx})} (B,Ffx) \xrightarrow{(1_B,\varphi)} (B,y)$.
- Any vertical arrow (u,ρ) can be factored as $(A,x) \xrightarrow{(u,1^\bullet_{Fux})} (A',Fux) \xrightarrow{(1^\bullet_{A'},\rho)} (A',x')$.
- And any double cell (α,Φ) can be factored as

$$
\begin{tikzcd}
(A,x) \arrow[r, "{(f,1_{Ffx})}"] \arrow[dd, "{(u,1^\bullet_{Fux})}"'] & (B,Ffx) \arrow[r, "{(1_B,\varphi)}"] \arrow[d, "{(v,1^\bullet_{F(vf)x})}"'] \arrow[dr, phantom, "{(1_v,1^\bullet_{Fv\varphi})}"] & (B,y) \arrow[d, "{(v,1^\bullet_{Fvy})}"] \\
{} \arrow[r, phantom, "{(\alpha,1_{(F\alpha)_x})}"] & (B',FvFfx) \arrow[r, "{(1_{B'},Fv\varphi)}"] \arrow[d, "{(1^\bullet_{B'},(F\alpha)_x)}"] & (B',Fvy) \arrow[dd, "{(1^\bullet_{B'},\lambda)}"] \\
(A',Fux) \arrow[r, "{(f',1_{F(f'u)x})}"] \arrow[d, "{(1^\bullet_{A'},\rho)}"'] \arrow[dr, phantom, "{(1^\bullet_{f'},1_{Ff'\rho})}"] & (B',Ff'Fux) \arrow[d, "{(1^\bullet_{B'},Ff'\rho)}"] \arrow[r, phantom, "{(1^\square_{B'},\Phi)}"] & {} \\
(A',x') \arrow[r, "{(f',1_{Ff'x'})}"'] & (B',Ff'x') \arrow[r, "{(1_{B'},\varphi')}"'] & (B',y')
\end{tikzcd}
$$

The Main Theorem

- There is a doubly lax cocone $F \overset{\lambda}{\Longrightarrow} \Delta \mathbb{G}\mathrm{r}\, F$ with the required universal property:

$$\lambda^* \colon \mathbf{DblCat}\left(\int_{\mathbb{D}} F, \mathbb{E}\right) \to \mathbb{L}\mathrm{C}\left(\int_{\mathbb{D}} F, \mathbb{E}\right)$$

  is an iso of double categories for all $\mathbb{E} \in \mathbf{DblCat}$.

- Furthermore, $\int_{\mathbb{D}}$ extends to a functor of $\mathbf{DblCat}$-categories

$$\mathrm{Hom}_v(\mathbb{D}, \mathbf{DblCat})_{d\ell} \to \mathbf{DblCat}/\mathbb{D}$$

  which is locally an isomorphism of double categories

$$\mathbb{H}\mathrm{om}_{d\ell}(F, G) \cong (\mathbf{DblCat}/\mathbb{D})\left(\int_{\mathbb{D}} F \to \mathbb{D}, \int_{\mathbb{D}} G \to \mathbb{D}\right).$$

Example: Semidirect Products

Let K and Q be groups and $\theta\colon Q \to \mathrm{Aut}(K)$. Then the double categorical version of the semidirect product $K \rtimes_\theta Q$ is the following doubly lax colimit:

- Let $\mathbb{H}B(Q)$ be the horizontal indexing double category;
- Define the vertical functor $D\colon \mathbb{H}B(Q) \to\!\!\!\!\!\!\!\bullet\;\;\; \mathbf{DblCat}$ on objects by $D(\bullet) = \mathbb{V}B(K)$
- On arrows, $D(x)(\bullet) = \bullet$ and $D(x)(k) = \theta_x(k)$ for $x \in Q$ and $k \in K$. (Since θ_x is an automorphism, this is a well-defined double functor.)
- Then the double category of elements, $\int_{\mathbb{H}B(Q)} D$, has double cells

$$
\begin{tikzcd}
(\bullet,\bullet) \arrow[r, "{(x,1_\bullet)}"] \arrow[d, "{(1^\bullet_\bullet,k)}"'] \arrow[dr, phantom, "{(1^\bullet_x,1_{\theta_x(k)})}"] & (\bullet,\bullet) \arrow[d, "{(1^\bullet_\bullet,\theta_x(k))}"] \\
(\bullet,\bullet) \arrow[r, "{(x,1_\bullet)}"'] & (\bullet,\bullet)
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
\bullet \arrow[r, "x"] \arrow[d, "k"'] \arrow[dr, phantom, "{(x,k)}"] & \bullet \arrow[d, "\theta_x(k)"] \\
\bullet \arrow[r, "x"'] & \bullet
\end{tikzcd}
$$

Colimits of vacant double groupoids

- Let $\mathbb{D}$ be a vacant double groupoid and $\mathbb{D} \xrightarrow{F} \mathbf{DblCat}$ a vertical double functor valued in vacant double groupoids.
- A double functor $L\colon \mathbb{X} \to \mathbb{Y}$ between vacant double groupoids has the *horizontal lifting property* if for any horizontal arrow $Lx \xrightarrow{\varphi} y$ in $\mathbb{Y}$, there is a horizontal arrow $x \xrightarrow{\psi} x'$ in $\mathbb{X}$ with $L(\psi) = \varphi$.
- If $F(v)$ has the horizontal lifting property for each vertical arrow v in $\mathbb{D}$, then the colimit double category $\int_{\mathbb{D}} F$ is again a vacant double groupoid.

Application I: Tricolimits in **2-Cat**

- For a 2-category $\mathcal{A}$ and a 2-functor $F\colon \mathcal{A} \to \mathbf{2\text{-}Cat}$, consider

$$\mathcal{A} \xrightarrow{F} \mathbf{2\text{-}Cat} \xrightarrow{\mathbb{V}} \mathbf{DblCat}_v$$

  and then apply $\mathbb{V}$ to obtain:

$$\mathbb{V}(\mathcal{A}) \xrightarrow{\mathbb{V}(\mathbb{V}\circ F)} \mathbb{V}(\mathbf{DblCat}_v) \xrightarrow{\text{incl}} \mathbb{Q}(\mathbf{DblCat}_v).$$

- Applying the double Grothendieck construction gives us

$$\int_{\mathbb{V}\mathcal{A}} \mathbb{V}(\mathbb{V} \circ F) = \mathbb{V} \int_{\mathcal{A}} F$$

  (as defined by Bakovic and Buckley)
- The functor $\mathbb{V}\colon \mathbf{2\text{-}Cat} \to \mathbf{DblCat}_v$ induces an isomorphism of 3-categories between **2-Cat** and its image in $\mathbf{DblCat}_v$.
- It follows that $\int_{\mathcal{A}} F$ is the **lax tricolimit** of F in **2-Cat**.

Application II: Categories of Elements

- For a functor $F\colon \mathbf{A} \to \mathbf{Set}$,

  $$\operatorname{colim} F = \pi_0 \mathrm{El}\,(dF),$$

  where

  $$\mathbf{A} \xrightarrow{F} \mathbf{Set} \xrightarrow{d} \mathbf{Cat}$$

  and $\mathrm{El}\,(dF)$ has objects (A, x) with $x \in F(A)$ and arrows $f\colon (A, x) \to (A', x')$ where $f\colon A \to A'$ with $F(f)(x) = x'$.
- This follows from the universal property of the elements construction as lax colimit by applying it to cones with discrete categories as vertex and using the adjunction $\pi_0 \dashv d$.

- We can apply the same paradigm to a functor $F \colon \mathcal{A} \to \mathbf{Cat}$ and use

$$\begin{tikzcd} \mathbf{Cat} \arrow[r, shift right, "\mathbb{V}"'] \arrow[r, phantom, "\perp"] & \mathbf{DblCat}_v \arrow[l, shift right, "\pi_0"'] \end{tikzcd}$$

where the π_0 is taken with respect to horizontal arrows and cells to obtain a quotient of the vertical category of a double category.

- It follows from our Main Theorem that $\pi_0 \int_{\mathbb{H}\mathbf{A}} \mathbb{Q}(\mathbb{V} \circ F)$ gives the **strict 2-categorical colimit** of F.

Application III: Lax Tricolimits in $\mathbf{DblCat}_v$

For a 2-functor $F\colon \mathbf{A} \to \mathbf{DblCat}_v$, $\int_{\mathbb{V}\mathbf{A}} \mathbb{V}(F)$ is the lax tricolimit of F in $\mathbf{DblCat}_v$.

Other Results and Work in Progress

- Describe the notion of fibration between double categories that characterizes the double functors of the form $\int_{\mathbb{D}} F \to \mathbb{D}$ and extend our results to a correspondence between suitable fibrations over $\mathbb{D}$ and (double pseudo) indexing functors $\mathbb{D} \xrightarrow{v} \mathbf{DblCat}$.
- Extend the construction and the correspondence to double pseudo indexing functors $\mathbb{D} \to \mathbb{Q}(\mathbf{DblCat}_v)$.

Thank you!