Colimits of Bicategories, Fibrations, Fractions and Filteredness*

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The Grothendieck construction associates a fibration $\int F$ over $\mathcal{C}$ to each pseudo-functor $F : \mathcal{C}^{\mathrm{op}} \to Cat$. A classic result that can be found in [7] is that the pseudo-colimit of F can be computed by localizing this Fibration at the cartesian arrows, and that when $\mathcal{C}$ is (pseudo)Filtered this localization can be constructed using a calculus of Fractions. These notions (Fibrations, Fractions and Filteredness) have been generalized to the context of bicategories [2, 10, 11, 4], and we show how they can be used, in a similar fashion to the one in [7], to compute (higher dimensional) colimits of diagrams of bicategories, indexed by bicategories.

The Grothendieck construction can be naturally generalized to dimension 2 [9, 1, 2]. In its most general studied case, for $\mathcal{B}$ a bicategory, it's a trihomomorphism

$$\int : [\mathcal{B}, \mathbf{Bicat}]_{\mathrm{weak}} \longrightarrow (\mathbf{Bicat}/\mathcal{B}) \qquad (\star)$$

from the tricategory of trihomomorphisms, trinatural transformations, trimodifications and perturbations, to the strict slice tricategory of bicategories over $\mathcal{B}$ (we use the covariant case throughout this work). Looking carefully at the construction, one can notice that it still makes sense when some of the considered data is *lax*. We use the construction described by Buckley [2] to *reverse engineer* a definition of *lax functor* and *lax natural transformation* in dimension 3. We obtain very similar definitions to those introduced in [6] or [8], but with some subtle changes.

This idea of *reverse engineering* definitions of lax structures in dimension 3, out of the 2-dimensional Grothendieck construction, gives good motivations for our definitions. It also highlights links between the definitions of:

- bicategories **and** lax functors between tricategories,
- pseudo-functors between bicategories **and** lax natural transformations between lax functors between tricategories,
  and so on...

by exploiting the dimensional shift inherent to the construction.

The functor $\int$ in $(\star)$ is known to be a local biequivalence [2], and we have extended this result to this larger *lax* setting. We obtain in particular that

$$\int : [\mathcal{B}, \mathbf{Bicat}]_{\mathrm{lax}}(F, G) \longrightarrow (\mathbf{Bicat}/\mathcal{B})^{\mathrm{co}} \left(\int F, \int G \right)$$

is a biequivalence whenever G is, for example, a trihomomorphism.

We then use this result to construct all types of bicategory-indexed colimits in **Bicat** (and hence in **Cat** too by applying π_0 left adjoint to the inclusion of categories into bicategories). Note that we allow our diagrams to be lax, so when the indexing bicategory is discrete we recover in particular the results from [3] with a different, conceptual proof. Using the key observation that $\int \Delta(\mathcal{X}) \simeq \mathcal{X} \times \mathcal{B}$, when $\Delta(\mathcal{X})$ is the constant strict functor at the bicategory $\mathcal{X}$, we apply this result to:

- compute conical pseudo-colimits directly with Buckley's result,
- compute conical lax-colimits directly with our generalization,

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- compute conical σ-colimits [5], which gives us any type of limits in **Cat**, with a more precise fine-tuning of the result.

Using the other key observation that $\int \mathbf{Cat}(F(-), \mathcal{X}) \simeq (F \downarrow \mathcal{X})$, where $F : \mathcal{B} \to \mathbf{Cat}$ is a pseudo-functor, $\mathcal{X}$ is a category and $(F \downarrow \mathcal{X})$ is a form of lax comma, we can also directly compute weighted pseudo-colimits in **Cat** (we could extend this to **Bicat** with substantially more tricategory theory).

This lays out a method to formulate the construction of colimits in higher dimensions. Checking the local equivalence of the construction, which is a crucial step, may be very technical but may also be studied more abstractly: we are working on a more fundamental approach by introducing several notions of *lax Kan extensions* pushing some ideas of Street [12] in dimension 3. Another crucial step, is that those constructions lead to the required colimits if and only if we know that the localizations exist. This lead us to focusing on a main known case of localization in dimension 2: the calculus of fractions.

We then explore the relations between fibrations, fractions and filteredness in bicategory theory. In a series of lemmas, we generalize some results from ordinary category theory [7] to dimension 2: the family of all arrows of a filtered bicategory admits a calculus of fractions, and if $p : \mathcal{E} \to \mathcal{B}$ is a fibration and $\mathcal{W}$ admits a calculus of fractions on $\mathcal{B}$, then the family of cartesian arrows above $\mathcal{W}$ admits a calculus of fractions.

Using these results and carefully checking that π_0 commutes with the bicategorical calculus of fractions [10, 11], we can compute filtered pseudo-colimits in **Cat** indexed by a bicategory by simply computing the ordinary one-dimensional calculus of fractions. As an application, we get a fomula for the hom-categories of a bicategory of fractions [10] as a filtered pseudo-colimit, which draws another link between filteredness and fractions.

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