A Tannakian approach to Grothendieck Galois theory

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Grothendieck Galois theory

We begin with

$$\begin{array}{c} \mathcal{C} \\ \Big\downarrow{\scriptstyle F} \\ \mathcal{E}ns_{<\infty} \end{array}$$

Under certain hypotheses we have a lifting of F

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\tilde{F}", "\approx"'] \arrow[d, "F"'] & \beta Aut(F)_{<\infty} \arrow[dl, "U"] \\ \mathcal{E}ns_{<\infty} & \end{tikzcd}$$

which is an equivalence of categories.

localic Galois theory (by Dubuc)

- We may omit the finiteness hypothesis on F.
- Now the group $Aut(F)$ doesn't yield the equivalence of categories.
- We construct instead a localic group $\ell Aut(F)$ whose points are the automorphisms of F.
- Under the finiteness hypothesis on F, $\ell Aut(F)$ is isomorphic to the group of its points.

Tannaka theory by Joyal-Street

We begin with

$$\begin{tikzcd} \mathcal{D} \arrow[d, "T"'] \\ Vec_{<\infty} \end{tikzcd}$$

Under certain hypotheses we have a lifting of T

$$\begin{tikzcd} \mathcal{D} \arrow[r, "\tilde{T}", "\approx"'] \arrow[d, "T"'] & Cmd_{<\infty}(End^{\vee}(T)) \arrow[dl, "U"] \\ Vec_{<\infty} & \end{tikzcd}$$

which is an equivalence of categories.

The generalization "$\mathcal{V}$ instead of *Vec*"

- $Vec \rightsquigarrow \mathcal{V}$ an arbitrary tensor category,
  $Vec_{<\infty} \rightsquigarrow \mathcal{V}_0$ subcategory of objects that have a dual.
- The definitions and constructions can be generalized.
- $End^{\vee}(T)$ is now a Hopf algebra in $\mathcal{V}$.

Note: It is an open problem if $\tilde{T}$ is an equivalence in this general case.

Considering $\mathcal{V} = s\ell$

$$\begin{tikzcd} \mathcal{C} \arrow[d, "F"'] \arrow[dd, bend left, "T"] \\ \mathcal{E}ns \arrow[d, "\ell"'] \\ s\ell_0 \end{tikzcd}$$

$$s\ell_0 = \{\ell X\}_{X \in \mathcal{E}ns}$$

$End^{\vee}(T)$: Hopf algebra in $s\ell \leftrightsquigarrow$ localic group.

Then we can compare the Galois construction $\ell Aut(F)$ with the Tannakian construction $End^{\vee}(T)$.

Main results (so far)

- We have a surjective locale morphism $\ell Aut(F) \to End^{\vee}(T)$.
- $Points(End^{\vee}(T)) = Aut(T) = Aut(F) = Points(\ell Aut(F))$.
- For the Grothendieck case, in which $\ell Aut(F)$ has enough points, so does $End^{\vee}(T)$ and therefore both constructions are isomorphic.
- In this case, since

$$\begin{array}{ccc} \text{action of the localic group} & \leftrightsquigarrow & \text{comodule of the Hopf} \\ \text{in a set } X & & \text{algebra in } \ell X, \end{array}$$

it follows that the Galois and Tannaka theorems are equivalent.

Thank you!