Implementing Double Categories in the Lean Proof Assistant

Zach Murray
Supervised by Dorette Pronk and Martin Szyld

Dalhousie University

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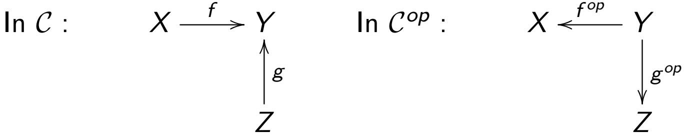
The Lean Proof Assistant

- The **Lean proof assistant** is a programming language where we can write proofs as programs
- If the program runs, then our proofs are correct
- But programming languages can be strict!
- Some forms of a definition fit the language better than others
- We will get a taste of why, and what it is like to write a usable definition in Lean by investigating double categories

Writing Usable Definitions in Lean

- Lean automatically proves equalities that are true by definition
- Extra work may be needed to automate other trivial equalities
- **Example**[1]: $(C^{op})^{op} = C$



  - True, but not definitionally (not automatic)
  - **Solution**: Two associativity axioms
    - $f \circ (g \circ h) = (f \circ g) \circ h$
    - $(f \circ g) \circ h = f \circ (g \circ h)$
- What similar issues have we encountered with double categories?

[1] Carette, J. (2022, September 22). *What I learned from formalizing Category Theory in Agda* [Talk]. Topos Institute Colloquium. `https://tinyurl.com/3mpbhmes`.

Internal Categories

- A **category internal to a category** A with pullbacks is a category formulated within A

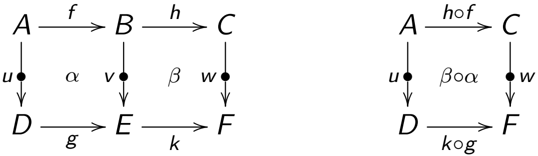
$$\mathbb{D}_1 \times_{t,s} \mathbb{D}_1 \xrightarrow{c} \mathbb{D}_1 \underset{t}{\overset{s}{\rightrightarrows}} \mathbb{D}_0 \quad (e: \mathbb{D}_0 \to \mathbb{D}_1)$$

- It consists of:
  - An object $\mathbb{D}_0 \in A$ of objects
  - An object $\mathbb{D}_1 \in A$ of arrows
  - Domain s and codomain t arrows
  - Composition arrow c
  - Identity-assigning arrow e
  - Axioms that make them interact like their category counterparts

Double Categories

- A **double category** consists of:
  - Objects $\mathbb{D}_{00}$
  - Vertical arrows $\mathbb{D}_{01}$
  - Horizontal arrows $\mathbb{D}_{10}$
  - Squares $\mathbb{D}_{11}$

  With category structures defining vertical and horizontal compositions of arrows and squares
  - $(\mathbb{D}_{00}, \mathbb{D}_{01})$ defining vertical composition of arrows
  - $(\mathbb{D}_{00}, \mathbb{D}_{10})$ defining horizontal composition of arrows
  - $(\mathbb{D}_{01}, \mathbb{D}_{11})$ defining horizontal composition of squares
  - ...etc



Double Categories as Internal Categories

- Double categories are equivalently categories internal to **Cat**

$$\begin{array}{ccccc} & & (\mathbb{D}_{10}, \mathbb{D}_{11}) & & (\mathbb{D}_{00}, \mathbb{D}_{01}) \\ & & \| & & \| \\ \mathbb{D}_1 \times_{t,s} \mathbb{D}_1 & \xrightarrow{c} & \mathbb{D}_1 & \overset{s}{\underset{t}{\rightrightarrows}} \; \overset{e}{\leftarrow} & \mathbb{D}_0 \end{array}$$

- We get:
  - $\mathbb{D}_0 = (\mathbb{D}_{00}, \mathbb{D}_{01})$ for vertical composition of arrows
  - $\mathbb{D}_1 = (\mathbb{D}_{10}, \mathbb{D}_{11})$ for vertical composition of squares
  - With the arrows in **Cat** of the internal category being constructed via the categories defining the compositions of a double category

Categories and Double Categories in Lean

- A category in Lean has:
  - A collection C_0 of objects
  - A family $\{C(X,Y) \mid X, Y \in C_0\}$ of collections of arrows
  - ...etc
- Define a double category as a category internal to **Cat**
  - For $\mathbb{D}_0$: $\mathbb{D}_{00}$ is a collection; $\mathbb{D}_{01}$ is represented by a family instead of a collection
  - For $\mathbb{D}_1$: $\mathbb{D}_{10}$ is a collection; $\mathbb{D}_{11}$ is represented by a family
- **Problem:** To construct the category $(\mathbb{D}_{01}, \mathbb{D}_{11})$ for horizontal composition of squares, we need $\mathbb{D}_{01}$ to be represented by a *collection* (single object $\mathbb{D}_{01}$), not a family of collections
  - Easily converting from collection-to-family and back may require a trick like with $(C^{op})^{op}$
  - What are some alternatives?

An Alternative

- A category is a category internal to **Set**
- Double Category: Category internal to (the category of categories internal to **Set**)
  Categories internal to **Set**:
  - $(\mathbb{D}_{00}, \mathbb{D}_{01})$ for vertical arrow composition
  - $(\mathbb{D}_{10}, \mathbb{D}_{11})$ for vertical square composition
  - $(\mathbb{D}_{01}, \mathbb{D}_{11})$ for horizontal square composition
- **Pro:** No conversion
  - $\mathbb{D}_{01}$ and $\mathbb{D}_{10}$ is a set of arrows, not a family of collections
  - $(\mathbb{D}_{01}, \mathbb{D}_{11})$ is a category internal to **Set**
- **Con:** Our categories are too small!

The Solution

- **Idea:** Use Lean's infinite hierarchy of classes
  - **Set** 0 is the category of sets
  - **Set** 1 is the category of proper classes
  - ...etc
- All categories can be represented as a category internal to **Set** u for some level u
- Double Category: Category internal to (the category of categories internal to **Set** u)
- Any size issues?
  - No, size can be selected using the levels as needed
- Any conversions?
  - $\mathbb{D}_{01}$ and $\mathbb{D}_{10}$ are classes
  - $(\mathbb{D}_{01}, \mathbb{D}_{11})$ is represented as an internal category in **Set** u
  - So... No!

More About Lean

- How do people figure out all these tricks for defining things??
- Who defined categories in Lean??
- **Answer:** The massive Lean community!
- Lean has a big library called **mathlib** with analysis, algebra, category theory, and much more
- Lean's community also has a very active Zulip instance with **over 6000 users** where they help users of all levels of experience
  - `https://leanprover.zulipchat.com/`
- In fact, it was with the community's help that we defined double categories in Lean[2]

[2] See `https://leanprover.zulipchat.com/#narrow/stream/217875-Is-there-code-for-X.3F/topic/double.20categories`.

The End

Thanks for listening!
Questions?