*where*

$$\Delta x = \frac{b-a}{n}$$

*and*

$$\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i) = \text{ \textit{midpoint of} } [x_{i-1}, x_i].$$

**Problem 168** (Exercise 12, page 377)**.** Use the Midpoint Rule with the $n = 4$ to approximate

$$\int_1^5 x^2 e^{-x} dx.$$

*Solution.*

Some quick properties of the definite integral:

$$\int_b^a f(x)dx = \int_a^b -f(x)dx$$

$$\int_a^a f(x)dx = 0$$

$$\int_a^b c dx = c(b-a), \text{ where } c \text{ is any constant.}$$

$$\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

$$\int_a^b c[f(x)]dx = c\int_a^b f(x)dx$$

$$\int_a^b [f(x) - g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$$

$$\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$$

## 5.3 The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus (FTC) links the two branches of calculus: differential calculus (this course mostly) and integral calculus (this chapter and Math 1010). It does so by describing how integration and differentiation are inverse processes. Also note that the FTC has two parts, usually just called Part 1 and Part 2, but the parts are often interchange in other textbooks.

The first part of the FTC deals with functions defined by an equation of the form

$$g(x) = \int_a^x f(t)dt$$

where $f$ is a continuous function on $[a, b]$ and $x$ varies between $a$ and $b$.

If $f$ is positive, we can think of $g(x)$ as the area under the graph of $f$ from $a$ to $x$, or the area "so far".

It turns out that $g$ is the antiderivative of $f$. This gives:

**Theorem 169** (The Fundament Theorem of Calculus, Part 1)**.** *If $f$ is continuous on $[a, b]$, then the function $g$ defined by*

$$g(x) = \int_a^x f(t)dt$$

*for $a \leq x \leq b$ is continuous on $[a, b]$ and differentiable on $(a, b)$, and $g'(x) = f(x)$.*

*Proof.* And discussion on page s 381-383.

**Problem 170** (Exercise 10, page 388)**.** Find the derivative of

$$g(x) = \int_0^x \sqrt{t^2 + 4}dt.$$

*Solution.*

**Problem 171** (Exercise 12, page 388)**.** Find the derivative of

$$G(x) = \int_x^1 \cos(\sqrt{t})dt.$$

*Solution.*

**Problem 172.** Find

$$\frac{d}{dx} \int_1^{x^4} \sec(t)dt.$$

*Solution.*

In the previous section, we saw that to work out the definite integral as the limit of Riemann sums is difficult. Part 2 of the FTC gives a much simpler method:

**Theorem 173** (The Fundamental Theorem of Calculus, Part 2)**.** *If $f$ is continuous on $[a, b]$, then*

$$\int_a^b f(x)dx = F(b) - F(a),$$

*where $F$ is any antiderivative of $f$, that is, a function such that $F' = f$.*

*Proof.* See pages 384-385.

Part 2 tell us that if we know an antiderivative $F$ of $f$ then we can evaluate $\int_a^b f(x)dx$ simply by subtracting the values of $F$ at the endpoints of the interval $[a, b]$.

**Problem 174** (Exercise 22, page 388)**.** Evaluate the integral

$$\int_0^1 (1 + \frac{1}{2}u^4 - \frac{2}{5}u^9)du.$$

*Solution.*

**Problem 175** (Exercise 26, page 388)**.** Evaluate

$$\int_\pi^{2\pi} \cos\theta d\theta.$$

*Solution.*

**Problem 176.** Evaluate

$$\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{6}{\sqrt{1-t^2}}dt.$$

*Solution.*

**Problem 177** (Exercise 25, page 397)**.** Evaluate the integral

$$\int_{-2}^{2} (3u+1)^2 du.$$

*Solution.*

## 5.4 Indefinite Integrals and the Net Change Theorem

In Section 5.3 we used the FTC, part 2, to evaluate the definite integral of a function. Now we look at indefinite integrals (those without limits of integration). This is the equivalent to 'derivatives as functions' as compared to derivatives at a point.

The notation $\int f(x)dx$ is going to be used for the antiderivative of $f$ and is called an *indefinite integral.*

$$\int (fx)dx = F(x) \text{ means } F'(x) = f(x).$$

For example,

$$\int x^2 dx = \frac{x^3}{3} + C.$$

*Note.* A definite integral $\int_a^b f(x)dx$ is a number, whereas an indefinite integral $\int f(x)dx$ is a function (or a family of functions).

**Table of Indefinite Integrals**

$$\int c \cdot f(x)dx =$$

$$\int [f(x) + g(x)]dx =$$

$$\int kdx =$$

$$\int x^n dx =$$

$$\int \frac{1}{x} dx =$$

$$\int e^x dx =$$

$$\int a^x dx =$$

$$\int \sin x dx =$$

$$\int \cos x dx =$$

$$\int \sec^x dx =$$

$$\int \csc^x dx =$$

$$\int \sec x \tan x dx =$$

$$\int \csc x \cot x dx =$$

$$\int \frac{1}{1+x^2} dx =$$

$$\int \frac{1}{\sqrt{1-x^2}} dx =$$

*Note.* We adopt the convention that when a formula for a general indefinite integral is give, it is valid only on an implied interval. For example, we write

$$\int \frac{1}{x^2} dx = -\frac{1}{x} + C$$

with the understanding that it is valid on the interval $(0, \infty)$ or $(-\infty, 0)$ only.

**Problem 178** (Exercise 14, page 397)**.** Find

$$\int (\csc^2 t - 2e^t)dt.$$

*Solution.*

**Problem 179** (Exercise 5, page 397)**.** Find

$$\int (x^2 + x^{-2} dx.$$

*Solution.*

**Problem 180.** Find

$$\int \frac{cos\theta}{sin^2\theta} d\theta.$$

*Solution.*

### 5.4.1 Applications

Part 2 of the FTC says that if $f$ is continuous on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

where $F$ is any antiderivative of $f$. This meant that $F' = f$, so we could rewrite this as

$$\int_a^b F'(x)dx = F(b) - F(a).$$

Consider $F'(x)$ as a rate of change.

**Theorem 181** (The Net Change Theorem)**.** *The integral of a rate of change is the net change:*

$$\int_a^b F'(x)dx = F(b) - F(a).$$

In terms of some of the applications we have already discussed:

- If the rate of growth of a population is $\frac{dn}{dt}$, then

  $$\int_{t_1}^{t_2} \frac{dn}{dt} dt = n(t_2) - n(t_1)$$

  is the net change in population during the time period from $t_1$ to $t_2$.

- If an object moves along a straight line with position function $s(t)$, then its velocity is $v(t) = s'(t)$, so

  $$\int_{t_1}^{t_2} v(t)dt = s(t_2) - s(t_1)$$

  is the net change of position or *displacement* of the particle during the time period from $t_q$ to $t_2$.

Also,

$$\int_{t_1}^{t_2} |v(t)|dt = \text{ total distance travelled.}$$

The acceleration of the object is $a(t) = v'(t)$, so

$$\int_{t_1}^{t_2} a(t)dt = v(t_2) - v(t_1)$$

is the change in velocity from time $t_1$ to time $t_2$.

## 5.5 The Substitution Rule

The antiderivative formulas we looked at still do not tell us how to do all integrals, here we introduce a rule that allows us to find integrals of functions such as

$$\int 2x\sqrt{1+x^2}dx.$$

Here we use

**Theorem 182** (The Substitution Rule)**.** *If $u = g(x)$ is a differentiable function whose range is an interval $I$ and $f$ is continuous on $I$, then*

$$f(g(x)g'(x)dx = \int f(u)du.$$

The Substitution Rule is functionally the inverse of the Chain Rule, when using the Substitution Rule we're making a transformation that simplifies the inside function allowing us to see how to integrate the outside function.

**Example 183.** Consider the integral

$$\int 2x\sqrt{1+x^2}dx.$$

Here we let $u = 1 + x^2$. Taking the derivative of each side with respect to $x$ we find

$$\frac{du}{dx} = 2x.$$

Multiplying each side by $dx$ gives $du = 2xdx$. We see now that our integrand is just $\sqrt{1+x^2}(2xdx) = \sqrt{u}\cdot du$. Thus our integral becomes

$$\int \sqrt{u}du = \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C.$$

However, we want our answer in terms of $x$, which is

$$\frac{2}{3}(1+x^2)^{\frac{3}{2}}.$$

**Problem 184.** Calculate
$$\int e^{6x}dx.$$

*Solution.*

**Problem 185.** Calculate
$$\int x^2x^3 + 5^9dx.$$

*Solution.*

**Problem 186.** Calculate
$$\int \frac{x}{(x^2+1)^2}dx.$$

*Solution.*

**Problem 187.** Calculate
$$\int \sec(2\theta)\tan(2\theta)d\theta.$$

*Solution.*

### 5.5.1 Definite Integrals

Substitution is a little trickier with definite integrals because the limits of integration are in terms of a particular variable.

There are two equivalent methods (meaning both give the same correct answer) for evaluating definite integrals with substitution:

1. evaluate the indefinite integral first, then use the FTC.
2. change limits of integration (by plugging them in to $u(x)$), then evaluate the new definite integral in terms of $u$.

The second method above is stated formally in the following theorem.

**Theorem 188** (The Substitution Rule for Definite Integrals)**.** *If $g'$ is continuous on $[a, b]$ and $f$ is continuous on the range of $u = g(x)$, then*

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du.$$

**Problem 189.** Evaluate

$$\int_0^7 \sqrt{4+3x}dx.$$

*Solution.*

**Problem 190.** Evaluate

$$\int_0^2 (x-1)^{25}dx.$$

*Solution.*

**Problem 191.** Evaluate

$$\int_0^1 x^2(1+2x^3)^5dx$$

*Solution.*

**Problem 192.** Evaluate

$$\int_0^{\frac{\pi}{2}} \cos x \sin(\sin x) dx.$$

*Solution.*

### 5.5.2 Symmetry

We can use the following theorem to simplify calculation of integrals of functions that possess symmetry.

**Theorem 193** (Integrals of Symmetry Functions)**.** *Supposed $f$ is continuous on $[-a, a]$.*

*(a) If $f$ is even (i.e. $f(-x) = f(x)$), then $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$.*

*(b) If $f$ is odd (i.e. $f(-x) = -f(x)$), then $\int_{-a}^{a} f(x)dx = 0$.*