MATH 2113 - Assignment 10

Not to be handed in

1. Let $(S, +, \cdot)$ be an arbitrary ring. Prove that the set of units in S form a group under $\cdot$.

2. If G is a group, prove that, $\forall a, b \in G$, $(a^{-1})^{-1} = a$, and $(a*b)^{-1} = b^{-1}*a^{-1}$.

3. Consider a square centered on the origin, with edges parallel to the axes. What are the rigid motions which leave the square unchanged? (This would include rotating or flipping the square over.) Two rigid motions, A, B may be multiplied giving $A * B$, if we first do B, and then do A. Derive the group multiplication table.

4. Prove that that the groups $(\mathbb{Z}_6, +)$ and $(\mathbb{Z}_7, \cdot)$ are isomorphic where $+$ and $\cdot$ are regular addition and multiplication.

5. Prove that if a graph has n vertices and every vertex has degree at least $\frac{n}{2}$ then the graph is connected.

6. An induced subgraph of a graph G is a subset of the vertices of G. The induced subgraph has an edge xy if and only if xy is an edge in G. Prove that a simple connected graph on at least 3 vertices has an induced subgraph isomorphic to a path on 3 vertices if and only if G is not a complete graph.

7. An edge-colouring of a graph G is a function which maps the edges of G to the natural numbers such that edges which share an endpoint must have different colours. The edge-chromatic number of a graph is equal to the fewest number of colours needed to provide a legal colouring of the edges.

a) Find the edge-chromatic number of the Petersen graph.
b) Find a good lower bound for the edge-chromatic number of a graph G.
c) Find a good upper bound for the edge-chromatic number of a graph G.

8. Describe a method to find a maximum matching in a tree. Explain why your solution is maximum rather than maximal.

9. A cut-edge of a connected graph G is an edge such that its removal will disconnect the graph.

a) Prove that if every edge of a connected graph is a cut-edge, the graph is a tree.
b) Draw a graph which has a cut-edge where every vertex has degree 3.

10. Let G be a graph where $V(G) = \{1, 2, ..., n\}$ and there is an edge between a and b iff $a|b$.

a) Which vertices have the highest/lowest degrees?
b) Given that there is a cycle of length $k > 3$, prove that there is a cycle of length $k - 1$.