MATH 2113 - Midterm Solutions

February 18

1. A bag of marbles contains 4 which are red, 4 which are blue and 4 which are green.

   a) How many marbles must be chosen from the bag to guarantee that three are the same colour?

   We can apply the pigeonhole principle where the pigeons are the marbles and the holes are the colours. Since there are 3 colours, to guarantee 3 of the same colour, we need $3(2)+1 = 7$ pigeons (marbles).

   b) If you draw three marbles at random from the bag, what is the probability that they are the same colour?

   It makes no difference what colour is drawn first. From that point, 3 marbles left are the same colour (of the 11 remaining) so the probability the second is the same colour as the first is $\frac{3}{11}$. For the third marble, we must pick one of the two remaining from the available 10 so this has probability $\frac{2}{10}$. Therefore, the probability of picking all three the same colour is $\frac{3}{11}\frac{2}{10} = \frac{3}{55}$.

   c) How many marbles must be chosen from the bag to guarantee that there is at least one of every colour?

   This time, we can see that it is possible to choose 8 where only two colours are represented (all red and blue for example) but any 9 marbles will contain all colours of marbles. Therefore, the minimum required is 9.

   d) If you draw three marbles at random from the bag, what is the probability that they are all different colours?

As before, the first marble drawn does not matter. To choose one that is of a different colour is $\frac{8}{11}$. Then there are only 4 marbles left of the colour not yet represented, so the probability of drawing one of them is $\frac{4}{10}$. Therefore, the probability of drawing marbles of all three colours is $\frac{16}{55}$.

2. For how many integers from 1 to 9999 is the sum of their digits equal to 9?

   For all such integers, we recognize that this is the same as solving the problem of counting the number of solutions to $x_1 + x_2 + x_3 + x_4 = 9$ where $x_i \geq 0$. We could write the digits x_1 through x_4 in order to create a number between 1 and 9999. Using the result from class, we get that there are $\binom{9+3}{9} = 220$ solutions. Therefore, there are 220 positive integers less than 10000 which have this property.

3. a) Define $P(A|B)$.

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

   b) Prove that if $P(A) = P(B)$ then $P(A|B) = P(B|A)$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A \cap B)}{P(A)} = P(B|A)$$

   c) Define what it means for A and B to be independent events.

   A and B are independent if they satisfy $P(A|B) = P(A)$, or equivalently, $P(A \cap B) = P(A) \cdot P(B)$.

   Suppose there are three suppliers of computer parts X, Y and Z. Where 5% of X's products are superior quality, 10% of Y's products are superior quality and 15% of Z's products are superior quality. A particular store gets 50% of its parts from X, 30% from Y and 20% from Z.

d) If a unit is purchased, what is the probability that it is superior quality?

We will let S be the even that a unit is of superior quality. Since the superior product must have come from one of the three suppliers, we can deduce that

$$\begin{aligned} P(S) &= P(S \cap X) + P(S \cap Y) + P(S \cap Z) \\ &= (0.05)(0.5) + (0.1)(0.3) + (0.15)(0.2) \\ &= (0.025) + (0.03) + (0.03) \\ &= 0.085 \end{aligned}$$

e) If a unit in the store is found to be superior quality, which supplier is most likely to have come from?

For each of the three suppliers, we calculate the appropriate conditional probabilities:

$$P(X|S) = \frac{P(X \cap S)}{P(S)} = \frac{0.025}{0.085} \doteq 0.2941$$

$$P(Y|S) = \frac{P(Y \cap S)}{P(S)} = \frac{0.03}{0.085} \doteq 0.3529$$

$$P(Z|S) = \frac{P(Z \cap S)}{P(S)} = \frac{0.03}{0.085} \doteq 0.3529$$

So, the superior quality unit is most likely from either Y or Z (each with the same probability).

4. a) Define the mathematical expression "n choose r".

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

b) Prove Pascal's identity:

$$\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}$$

As seen in class, there are at least 2 or 3 ways to prove this identity. I will present my favourite here:

Suppose there are $n+1$ people in a room (one of whom is Bob). We want to find out how many ways there are to make a group of r of them. Of course, this is equal to $\begin{pmatrix} n+1 \\ r \end{pmatrix}$. We could also count the number of ways as follows: If we make a group without Bob, there are $\begin{pmatrix} n \\ r \end{pmatrix}$ combinations. If we include Bob, we must choose $r-1$ more from the remaining n giving $\begin{pmatrix} n \\ r-1 \end{pmatrix}$ combinations. Since every group either contains Bob or it doesn't, we must have counted exactly the same number of combinations. Therefore,

$$\begin{pmatrix} n+1 \\ r \end{pmatrix} = \begin{pmatrix} n \\ r \end{pmatrix} + \begin{pmatrix} n \\ r-1 \end{pmatrix}$$

5. Let $D = \{1, 2, 3, ..., 52\}$ and define $s : D \to D$ by

$$s(x) = \begin{cases} 2x - 1 & \text{if } 1 \leq x \leq 26 \\ 2x - 52 & \text{if } 27 \leq x \leq 52 \end{cases}$$

a) Is s a well defined function?

We can easily check that s is defined for all $x \in D$ and that for $1 \leq x \leq 26$, $1 \leq s(x) \leq 51$. Also, for $27 \leq x \leq 52$ we get that $2 \leq s(x) \leq 52$. Therefore, s is well defined.

b) Is s one to one?
c) Is s onto?

If $s(x)$ is odd then we can calculate that $x = \frac{s(x)+1}{2}$ and if $s(x)$ is even then we can calculate that $x = \frac{s(x)+52}{2}$. Since this defines the inverse of s, we have found the inverse and conclude that s is one to one and onto.

d) Find an expression for $s \circ s$.

We simply apply the function twice. Since $s(x)$ could be more or less than 26, we are going to get 4 cases instead of just two.

$$(s \circ s)(x) = \begin{cases} 4x - 3 & \text{if } 1 \leq x \leq 13 \\ 4x - 54 & \text{if } 14 \leq x \leq 26 \\ 4x - 105 & \text{if } 27 \leq x \leq 39 \\ 4x - 156 & \text{if } 40 \leq x \leq 52 \end{cases}$$

6. Let $D = \{1, 2, 3, ..., 52\}$ and let c be a constant such that $1 \leq c \leq 52$. Define $f_c : D \to D$ by

$$f_c(x) = \begin{cases} x - c & \text{if } x > c \\ x + 52 - c & \text{if } x \leq c \end{cases}$$

a) Prove that f_c is a well defined function.

f_c is defined for all x. In each of the two cases $1 \leq x - c \leq 52$ when $x > c$ and $1 \leq x + 52 - c \leq 52$ when $x \leq c$.

b) Prove that f_c is one to one for all c.

As we did in question 5, each case is a linear function which will be one to one. But it could be the case that $f_c(x_1) = f_c(x_2)$ if $x_1 > c$ and $x_2 \leq c$. But in this case we get that

$$\begin{aligned} f_c(x_1) &= f_c(x_2) \\ x_1 - c &= x_2 - c + 52 \\ x_1 &= x_2 + 52 \end{aligned}$$

This of course is impossible since the xs are chosen from D which only has values from 1 to 52. Therefore, it must be the case that f_c is one to one.

c) Prove that f_c is onto for all c.

Since f_c is a map between sets of the same size and f_c is also one to one, we can conclude that it must also be onto.

d) Find an expression for f_c^{-1}.

$$f_c^{-1}(x) = \begin{cases} x + c & \text{if } x \leq 52 - c \\ \\ x - 52 + c & \text{if } x > 52 - c \end{cases}$$

To double check you could compute $(f_c^{-1} \circ f_c)$ to show that it is the identity, though this is not required.