MATH 2113 - Assignment 10

Not to be handed in

1. Let $(S, +, \cdot)$ be an arbitrary ring. Prove that the set of units in S form a group under $\cdot$.

Let U be the set of units in S. To show this is a group under $\cdot$, we must show it is closed, associative, inverses exist and the identity exists.

Clearly, it must be associative since $\cdot$ is associative in S. Also, the identity is a unit, so it is also in U. By the definition of a unit, it's inverse is also a unit and hence also in U. So, we must only show that it is closed.

Let $a, b \in U$. Then a^{-1}, b^{-1} are also in U. Now consider the element ab. We know that

$$(ab)(b^{-1}a^{-1}) = a(bb^{-1})a^{-1} = aa^{-1} = 1$$

so ab is also a unit and hence in U. Therefore, U is closed and hence forms a group.

2. If G is a group, prove that, $\forall a, b \in G$, $(a^{-1})^{-1} = a$, and $(a*b)^{-1} = b^{-1}*a^{-1}$.

This was done in class and follows the same technique as question 1.

3. Consider a square centered on the origin, with edges parallel to the axes. What are the rigid motions which leave the square unchanged? (This would include rotating or flipping the square over.) Two rigid motions, A, B may be multiplied giving $A * B$, if we first do B, and then do A. Derive the group multiplication table.

This problem is the same as the example done in class with a triangle instead of a square.

4. Prove that that the groups $(\mathbb{Z}_6, +)$ and $(\mathbb{Z}_7, \cdot)$ are isomorphic where $+$ and $\cdot$ are regular addition and multiplication.

We can start by noticing that 1 is the identity of $(\mathbb{Z}_6, +)$ and 0 is the identity of $(\mathbb{Z}_7, \cdot)$. Therefore, our isomorphism must have that $f(0) = 1$. We can also note that 3 is the only element in $(\mathbb{Z}_6, +)$ which has order 2 while 6 is the only element of $(\mathbb{Z}_7, \cdot)$ which has order 2. Therefore, $f(3) = 6$. Following in this manner, we can determine that $f(1) = 5, f(2) = 4, f(4) = 2, f(5) = 3$ will provide an appropriate isomorphism.

5. Prove that if a graph has n vertices and every vertex has degree at least $\frac{n}{2}$ then the graph is connected.

Assume for a contradiction that the graph is disconnected. Then, there are 2 vertices x, y which are in different components (say S and T). Since every vertex has degree at least $\frac{n}{2}$, we know that $|S| \geq \frac{n}{2} + 1$ and $|T| \geq \frac{n}{2} + 1$. But then we find that $n \geq |S| + |T| \geq n + 2$ which is a contradiction. Therefore, the graph is connected.

6. An induced subgraph of a graph G is a subset of the vertices of G. The induced subgraph has an edge xy if and only if xy is an edge in G. Prove that a simple connected graph on at least 3 vertices has an induced subgraph isomorphic to a path on 3 vertices if and only if G is not a complete graph.

For the first direction, assume that G is a complete graph. Then every induced subgraph on 3 vertices must be K_3. Therefore, there is no induced subgraph isomorphic to P_3.

For the other direction, assume that G is not a complete graph. Then we must be able to find two vertices x and y which are not adjacent. Also, we know there is a path from x to y in G since it is connected. Let z be the first vertex along this path which is not adjacent to x (this could be y) and let x_1 be the vertex just before z along this path. Then, by construction, we know that the induced subgraph on x, x_1 and z is isomorphic to P_3. This completes the proof.

7. An edge-colouring of a graph G is a function which maps the edges of G to the natural numbers such that edges which share an endpoint must have different colours. The edge-chromatic number of a graph is equal to the fewest number of colours needed to provide a legal colouring of the edges.

a) Find the edge-chromatic number of the Petersen graph.

The edge-chromatic number is 4. The upper bound is given by the colouring that was shown in class. You can show that this is also a lower bound by starting with a colouring of the outside 5-cycle and notice that there is really only one case to consider. You will quickly be forced to use a 4th colour.

b) Find a good lower bound for the edge-chromatic number of a graph G.

As we saw in class, the best possible lower bound is the maximum degree of the graph.

c) Find a good upper bound for the edge-chromatic number of a graph G.

We also saw in class that one less than twice the maximum degree is a reasonable upper bound. The best possible upper bound is the maximum degree plus 1.

8. Describe a method to find a maximum matching in a tree. Explain why your solution is maximum rather than maximal.

The important observation here is that every tree is also a bipartite graph since it contains no odd cycles. We can then use our result from class involving augmenting paths to construct a maximum matching.

9. A cut-edge of a connected graph G is an edge such that its removal will disconnect the graph.

a) Prove that if every edge of a connected graph is a cut-edge, the graph is a tree.

Clearly, if every edge is a cut-edge, then there can be no cycles in the graph. Since the graph is connected, by definition it must be a tree.

b) Draw a graph which has a cut-edge where every vertex has degree 3.

I will leave this for you to solve.

10. Let G be a graph where $V(G) = \{1, 2, ..., n\}$ and there is an edge between

a and b iff $a|b$.

a) Which vertices have the highest/lowest degrees?

Clearly, any prime between $\frac{n}{2}$ and n can only be adjacent to 1 and hence has degree 1. Also, every vertex has degree at least 1 since vertex 1 is a neighbour of everyone. This also implies that 1 has the highest degree of $n-1$.

b) Given that there is a cycle of length $k > 3$, prove that there is a cycle of length $k-1$.

First we note that if there is a cycle of length k, there must be a cycle of length k passing through the vertex 1. If we are given any cycle not passing through 1, we can short-cut any vertex of the cycle by instead going to 1 then going on. That is to say, for any three vertices $x - y - z$ we could also have $x - 1 - z$. Now, given a path of length k passing through 1, we can short-cut any vertex adjacent to 1 to construct a cycle of length $k-1$. That is, if we have $x - y - 1 - z$ we could also have $x - 1 - z$. Note that we implicitly used the fact that $k > 3$.