MATH 2113 - Assignment 2 Solutions

Due: Jan 21

6.2.27 - Since we want to write $n = p_2^{k_2} p_2^{k_2} \cdot p_m^{k_m}$ as a product of two positive numbers without a common factor it must be of the form $n = a \cdot b$ where a contains some subset of the primes which divide n and b contains all the others. Since a and b cannot have a common factor, no prime can be a divisor of both. Therefore, we simply must choose which primes divide a. For each prime, we have two choices - if it divides a or it does not. Clearly, b must then be the product of the rest of the primes and is therefore determined. So, the number of ways to choose such an a is 2^m since there are m different primes.

In the case when order does not matter, we note that we have simply counted every pair exactly twice. Therefore, when order doesn't matter, there are only half as many combinations. Therefore, there are 2^{m-1} ways n can be written as such a product.

6.2.41 - Assume for the base case that the operation only has one step and that it can be done in n_1 ways. Clearly, this gives us a total of n_1 ways in which the operation can be done.

Now we assume that an operation with k steps, where step i can be performed n_i ways, can be performed in a total of $n_1 n_2 \cdots n_k$ ways.

Now, in an operation with $k + 1$ steps (and the last step can be performed in n_{k+1} ways) we can apply the induction hypothesis to determine that the first n steps can be performed in $n_1 n_2 \cdots n_k$ ways. The last step can be performed in n_{k+1} ways regardless of the previous steps. So, for any combination of the first k steps, we have n_{k+1} ways to proceed, so in total we have $n_1 n_2 \cdots n_k n_{k+1}$ ways to perform the entire operation as desired.

6.3.2 - Using the multiplication rule and addition rule, we find that there must be 16^3 strings of 3 digits, 16^2 strings of 2 digits and 16 singleton digits. Therefore, there are $16^3 + 16^2 + 16 = 4368$. To count the number of strings that have between 2 and 5 hexadecimal digits we simply add

$16^5 + 16^4 + 16^3 + 16^2 = 1118464$.

6.3.22 - By dividing, we can quickly determine that there are 500 numbers which are multiples of 2 and 111 numbers which are multiples of 9 between 1 and 1000. However, there are 55 numbers which are multiples of 18 (multiples of both 2 and 9) and so by the inclusion/exclusion rule we determine that there are 500+111-55=556 numbers which are multiples of 2 or 9.

Since there are 1000 numbers in the range, if we were to choose one at random, the probability that it is a multiple of 2 or 9 is $\frac{556}{1000} = \frac{139}{250}$.

Since there are 556 numbers which are multiples of 2 or 9, then the rest of the numbers in the range are not multiples of either. Therefore, there are 1000-556=444 of them.

6.4.7 - To choose 7 from a group of 13 we get $\binom{13}{7} = 1716$ different combinations.

To select 4 women from the 7 available and 3 men from the 6 available, we apply the multiplication rule to get $\binom{7}{4}\binom{6}{3} = 35 \cdot 20 = 700$ different combinations.

There is exactly $\binom{7}{7} = 1$ way to choose a team consisting entirely of women, so all the rest must contain at least one man. Therefore, there are 1716-1=1715 different combinations.

Using the addition rule, we simply find the sum of the ways to choose a team that contains 0,1,2 and 3 women. Since there are only 6 men, it is impossible to make a team of 7 without at least one woman. Therefore, there are exactly

$$\begin{aligned}\binom{7}{3}\binom{6}{4} + \binom{7}{2}\binom{6}{5} + \binom{7}{1}\binom{6}{6} &= (35)(15) + (21)(6) + (7)(1) \\ &= 525 + 126 + 7 \\ &= 658\end{aligned}$$

For part (c), if one of the two members is on the team, the other cannot be, so there are $\binom{11}{6} = 462$ ways to choose the rest of the team. If neither is on the team, then there are $\binom{11}{7} = 330$ ways to choose the team. In total, that gives us $462 + 462 + 330 = 1254$ ways to make the team. Finally, for part (d), we note that if two members are both on the team, there are $\binom{11}{5} = 462$ ways to choose the rest of the members while if neither is on the team, there are $\binom{11}{7} = 330$ ways to choose the team. So, in total, there are $462 + 330 = 792$ combinations for such a team.

6.4.11 (e) - To determine the number of hands which form a flush, we must select a suit and then choose 5 cards from that suit. We must be careful however and not count the hands that would make a straight flush or royal flush.

There are 4 suits to choose from. Then we note that there are $\binom{13}{5} = 1287$ ways to choose 5 cards from the set of 13 of that suit. Finally, we note that there are 10 different hands that would also make a straight flush or royal flush in a particular suit. So, in total there are 4(1287-10) = 5108 different ways to get a flush. Now, to calculate the probability of getting such a hand we divide by the total number of hands which is $\binom{52}{5} = 2598960$. This makes the probability $\frac{5108}{2598960} = 0.0019654...$

6.4.25 (a) - First we can write the prime factorization of 210 as (2)(3)(5)(7). To write this as a product of two positive integers, when we pick one of them, the other will be determined. In this case the pairs are: $\{1 \cdot 210, 2 \cdot 105, 3 \cdot 70, 5 \cdot 42, 7 \cdot 30, 6 \cdot 35, 10 \cdot 21, 14 \cdot 15\}$.

(d) - We note that to select a number which divides n, we must select a set of primes from the k which divide n. To select such a set, we must choose to include p_i or not for each prime. This gives us 2 choices for each prime and thus 2^k choices all together. Of course, each pair can be written in two different ways ($a \cdot b$ and $b \cdot a$) so we will have counted all pairs twice since order doesn't matter. So, in total, n can be written as a product of two positive integers in 2^{k-1} ways.

6.5.7 - If we have a solution to the problem $x_1 + x_2 + ... + x_n = m$ where x_i is non-negative then there is a unique n-tuple $(y_1, y_2, ..., y_n)$ where $y_i = x_1 + x_2 + ... + x_i$. Therefore, to count the number of solutions to the first question, we can instead count the number of n-tuples described. We know that $0 \leq y_1 \leq y_2 \leq ... \leq y_n \leq m$. One way to choose such a set is just to pick n numbers in the range 0 to m (inclusive) then arrange them from smallest to largest. Since we allow repetitions, the number of ways to select such a set is $\begin{pmatrix} m+n-1 \\ m \end{pmatrix}$

6.5.12 - Using the previous question, we find $m = 30$ and $n = 4$ so we get $\begin{pmatrix} 33 \\ 30 \end{pmatrix} = 5456$ different solutions.

6.5.15 - Again, we are allowing repetition, so the total number of combinations will be $\begin{pmatrix} 30+12-1 \\ 12 \end{pmatrix} = \begin{pmatrix} 41 \\ 12 \end{pmatrix} = 7898654920$. Of course, since there are 30 kinds of balloons and we are only choosing 12, it is impossible for a set to have at least one of each kind. Therefore, the probability is 0.