MATH 2113 - Assignment 3

Due: Jan 28

6.6.4 - Using the definition, we find that

$$\begin{aligned}\binom{n}{3} &= \frac{n!}{3!(n-3)!} \\ &= \frac{n(n-1)(n-2)(n-3)!}{3!(n-3)!} \\ &= \frac{n^3-3n^2+2n}{6}\end{aligned}$$

6.6.13 - Using the definition we can calculate

$$\begin{aligned}\binom{n}{r+1} &= \frac{n!}{(r+1)!(n-r-1)!} \\ &= n!\left(\frac{1}{(r+1)r!}\right)\left(\frac{n-r}{(n-r)(n-r-1)!}\right) \\ &= \frac{n-r}{r+1}\left(\frac{n!}{r!(n-r)!}\right) \\ &= \frac{n-r}{r+1}\binom{n}{r}\end{aligned}$$

6.6.20 - For a base case, we consider n=0. Since

$$\binom{m}{0} = 0 = \binom{m+0+1}{0}$$

we find that the statement is true for this case. Then, for $n = k$ the inductive hypothesis becomes

$$\binom{m}{0} + \binom{m+1}{1} + ... + \binom{m+k}{k} = \binom{m+k+1}{k}$$

Finally, we would like to prove the case where $n = k + 1$:

$$\begin{aligned}
& \binom{m}{0} + \binom{m+1}{1} + ... + \binom{m+k}{k} + \binom{m+k+1}{k+1} \\
= & \binom{m+k+1}{k} + \binom{m+k+1}{k+1} \\
= & \binom{m+k+2}{k+1} \\
= & \binom{m+(k+1)+1}{(k+1)}
\end{aligned}$$

Which is what is desired.

6.6.21 - We start be using the definition to get

$$\binom{p}{r} = \frac{p!}{r!(p-r)!}$$

Now, we know that the numerator is a multiple of p since we could write it as $p(p-1)!$. Also, the denominator is a product of integers all of which are less than p. Since p cannot be divided by any integer less than itself, we can determine that $\frac{(p-1)!}{r!(p-r)!} = k$ is also an integer. But then we have that

$$\binom{p}{r} = \frac{p(p-1)!}{r!(p-r)!} = pk$$

Which implies that it is indeed a multiple of p. Equivalently, p divides the quantity.

6.7.6 - Using the binomial theorem we can calculate:

$$\begin{aligned}
(u^2 - 3v)^4 &= \sum_{k=0}^{4} \binom{4}{k} (u^2)^{4-k}(-3u)^k \\
&= \binom{4}{0} (u^2)^4(-3)^0 + \binom{4}{1} (u^2)^3(-3)^1 + \binom{4}{2} (u^2)^2(-3)^2 \\
&+ \binom{4}{3} (u^2)^1(-3)^3 + \binom{4}{4} (u^2)^0(-3)^4 \\
&= (1)u^8 + (4)u^6(-3v) + (6)u^4(9v^2) + (4)u^2(-27v^3) + (1)(81v^4) \\
&= u^8 - 12u^6v + 54u^4v^2 + -108u^2v^3 + 81v^4
\end{aligned}$$

6.7.25 - Clearly, we can rewrite the formula and apply the binomial theorem as follows:

$$\sum_{i=0}^{m} \binom{m}{i} 4^i = \sum_{i=0}^{m} \binom{m}{i} (1)^{m-i}(4)^i = (1+4)^m = 5^m$$

6.8.3 - a) Since A and B are mutually exclusive, we know that $P(A \cap B) = 0$. Therefore, we apply the general union formula to get $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.4 + 0.2 - 0 = 0.6$.

b) First we note that since $A \cup B \cup C = S$ we have $P(A \cup B \cup C) = 1$. Then if we apply the general union formula, we get

$$1 = P(A \cup B \cup C) = P(A \cup B) + P(C) - P((A \cup B) \cap C)$$

We now use result of part a) and the fact that $P((A \cup B) \cap C) \geq 0$ to arrive at

$$1 \leq 0.6 + P(C)$$

Therefore, we conclude that $P(C) \geq 0.4$. Hence, it is impossible for $P(C) = 0.2$.

6.8.17 - We start by noting that there are $\binom{5}{2} = 10$ ways to pick two balls from the urn. This table summarizes the outcomes, their probabilities and sum appearing on the balls.

Outcome	Number of Outcomes	Probability	Payoff
$\{1,2\}$	2	$p_0 = \frac{1}{5}$	$a_0 = 3$
$\{1,8\}$	2	$p_1 = \frac{1}{5}$	$a_1 = 9$
$\{2,2\}$	1	$p_2 = \frac{1}{10}$	$a_2 = 4$
$\{2,8\}$	4	$p_3 = \frac{2}{5}$	$a_3 = 10$
$\{8,8\}$	1	$p_4 = \frac{1}{10}$	$a_4 = 16$

Now, we apply the formula for expected values to get:

$$\sum_{i=0}^{4} p_i a_i = \frac{1}{5}(3) + \frac{1}{5}(9) + \frac{1}{10}(4) + \frac{2}{5}(10) + \frac{1}{10}(16) = \frac{42}{5}$$

Therefore, the expected sum of the numbers printed on the balls is $\frac{42}{5} = 8.4$.

6.9.2 - If we rearrange the definition for conditional probability to solve for $P(X \cap Y)$, we get

$$P(X \cap Y) = P(Y) \cdot P(X|Y) = \left(\frac{1}{4}\right)\left(\frac{1}{3}\right) = \frac{1}{12}$$

6.9.14 - a) There are 180 parts altogether, and 100 are from the first factory, so the probability of choosing one from the first factory is $\frac{5}{9}$.

b) Clearly, the rest of the parts must be from the second factory, so the probability is $1 - \frac{5}{9} = \frac{4}{9}$.

c) We can use the addition rule along with the results from the first two parts. There is a $\frac{5}{9}$ probability that a chosen part is from the first factory and a $\frac{1}{50}$ probability that it is defective. Therefore, the probability of both of these things occurring is $\frac{5}{9}\frac{1}{50} = \frac{1}{90}$. We can apply the same logic for the parts from the second factory to get $\frac{4}{9}\frac{1}{20} = \frac{1}{45}$. Finally, we apply the addition rule (since a part can be from one factory, but not both) to get the probability of defective part to be $\frac{1}{90} + \frac{1}{45} = \frac{1}{30}$.

d) Using the definition of conditional probability we let X be the event that the part came from the first factory and let Y be the event that the part is defective. Now, $P(X \cap Y) = \frac{1}{90}$ and $P(Y) = \frac{1}{30}$. Therefore,

$$P(X|Y) = \frac{P(X \cap Y)}{P(Y)} = \frac{\frac{1}{90}}{\frac{1}{30}} = \frac{1}{3}$$