MATH 2113 - Assignment 4 Solutions

Due: Feb 9

7.1.3 c) This is not a function since there are two arrows pointing out of '4'.
d) This is a function since every element of X has a unique arrow associated with it.
e) This is not a function since it is not defined for all elements of X. There is no arrow associated with '2'.

7.1.8 - These functions are not the same. We can consider the element '0' in the domain: $H(0) = \lfloor 0 \rfloor + 1 = 0 + 1 = 1$ while $K(0) = \lceil 0 \rceil = 0$. Since the two functions do not give the same result, they cannot be equal.

7.1.41 - We would like to prove that $f(A \cup B) = f(A) \cup f(B)$ where $f(A)$ is the image of the set A with respect to the function f.

First, if $y \in f(A \cup B)$ then we have that there exists an $x \in A \cup B$ such that $f(x) = y$. Therefore we know that $x \in A$ or $x \in B$. But then we find that $y \in f(A)$ or $y \in f(B)$. We conclude that $y \in f(A) \cup f(B)$.

For the other direction we start with an element $y \in f(A) \cup f(B)$. This tells us that $y \in f(A)$ or $y \in f(B)$. But this tells us that there is some x in either A or B such that $f(x) = y$. In particular, $x \in A \cup B$ so we conclude that $y \in f(A \cup B)$.

Therefore the two sets are equal and we find that $f(A \cup B) = f(A) \cup f(B)$.

7.2.8 - a) H is not one-to-one because two elements of the domain map to the same element of the co-domain: $H(b) = H(c) = y$.
H is not onto since there is no element in the domain ($n \in X$) for which $H(n) = z$.

b) K is one-to-one because we can quickly check that every element in the co-domain has at most one arrow associated with it.
K is not onto however, since there is no arrow associated with $z \in Y$.

7.2.19 - We are given the function $f(x) = \frac{x+1}{x-1}$.

$f(x)$ is one-to-one: To prove this we assume that $f(x_1) = f(x_2)$.

$$\begin{aligned}
f(x_1) &= f(x_2) \\
\frac{x_1+1}{x_1-1} &= \frac{x_2+1}{x_2-1} \\
(x_1+1)(x_2-1) &= (x_2+1)(x_1-1) \\
x_1x_2 - x_1 + x_2 - 1 &= x_1x_2 - x_2 + x_1 - 1 \\
2x_2 &= 2x_1 \\
x_2 &= x_1
\end{aligned}$$

Since this implies that $x_1 = x_2$ we have proven f is one-to-one.

$f(x)$ is not onto. If we consider the element '1' of the co-domain we find that if $f(x) = 1$ then $x + 1 = x - 1$ which leads to 2=0 giving a contradiction. Therefore, there is no element of the domain for which $f(x) = 1$. We conclude that f is not onto.

7.2.23 - a) D is not one-to-one. A counterexample would be a pair of strings which give the same result. Consider $D(11) = 2 = D(1101)$.
b) D is onto. To prove this, we must demonstrate a string s which gives $D(s) = n$ for every integer n. First, if $n = 0$ we can use the string '10'. If n is a positive number, we let s be a string of n 1s. If n is negative, we let s be a string of $|n|$ 0s. In each case, $D(s) = n$. Therefore, D is onto.

7.3.8 - We can split up the set T into disjoint subsets as follows: $\{1,9\}$, $\{2,8\}$, $\{3,7\}$, $\{4,6\}$, $\{5\}$ Each of the first four sets represent the different ways for a pair of numbers to have a sum of 10. Since there are 5 sets, the pigeon hole principle states that we must choose 6 numbers before we are guaranteed to have two from the same set. Therefore, there exists a set of 5 numbers where no pair sums to 10. Such a set would be $\{1,2,3,4,5\}$.

7.3.11 - We can again divide our set up into two parts: $\{1,3,5,...,2n-1\},\{2,4,6,...,2n\}$ Each set has exactly n elements. The pigeon hole principle states that if we choose $n+1$ elements, we must have at least 1 element from each set. Since the second part of our partition contains only even numbers, we are guaranteed to have chosen an even number.

7.4.6 - We are given that $F(x) = 3x$ and $G(x) = \lfloor \frac{x}{3} \rfloor$. Therefore, $F \circ G$ is defined by $F(G(x)) = F(\lfloor \frac{x}{3} \rfloor) = 3\lfloor \frac{x}{3} \rfloor$. We can also calculate that $G \circ F$ is defined by $G(F(x)) = G(3x) = \lfloor \frac{3x}{3} \rfloor = \lfloor x \rfloor$.

We can show that $F \circ G \neq G \circ F$ by finding a real number for which they don't agree. In particular: $F(G(1)) = F(0) = 0$ while $G(F(1)) = G(3) = 1$.

7.4.14 - Given the bijection $f : X \to Y$ and its inverse $f^{-1} : Y \to X$, we want to prove that $f \circ f^{-1} = i_Y$.

First, we consider an arbitrary element $y \in Y$. Then we consider $f(f^{-1}(y))$. Since f is a bijection, f^{-1} is also a well-defined bijection where $f^{-1}(y) = x$ if and only if $f(x) = y$. Therefore, $f(f^{-1}(y)) = f(x) = y$. We also note that $i_Y(y) = y$ so we conclude that $f \circ f^{-1}$ and i_Y agree for all values and they are equal functions.