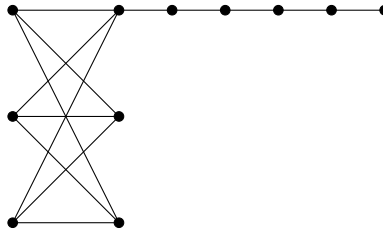


MATH 2113 - Assignment 8

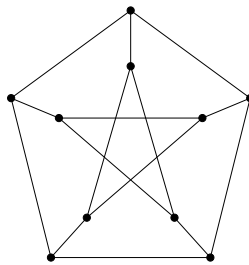
Due: Mar 21

1. Let n be a large number (say $n > 100$). Describe how to construct a connected non-planar graph which has n vertices and as few edges as possible.

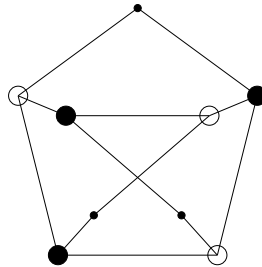
We know from Kuratowski's Thm. that we must have a subgraph that is homeomorphic to K_5 or $K_{3,3}$. To get the minimum number of edges, we note that K_5 has 5 vertices and 10 edges while $K_{3,3}$ has 6 vertices and 9 edges. Therefore, it would seem that using $K_{3,3}$ would be the better choice. Every time we add one more vertex, we must add at least one more edge to keep the graph connected (even when we sub-divide an edge). Therefore, if there are n vertices, we know we must have at least $n + 3$ edges. The following graph shows how this can be done by extending this path until the correct number of vertices are represented:



2. Prove that the Petersen graph (shown below) is not planar by finding a subgraph that is homeomorphic to $K_{3,3}$ or K_5 .



It is impossible to find a subgraph homeomorphic to K_5 since we would require a vertex of degree 4. The following subgraph is homeomorphic to $K_{3,3}$. The large solid vertices represent one partition, the large hollow vertices the other. The small vertices show the subdivided edges.

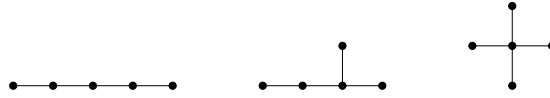


3. (page 721) 11.5.6

There are probably many ways to construct such a tree. One example would be to let the vertices be all the integers. Then have an edge between two integers if their difference is 1. Clearly, this makes an infinite path going in two directions. The graph is connected and has no cycles, but every vertex has degree 2.

4. (page 722) 11.5.30

Here are all the possible trees on 5 vertices:



5. Prove that the complement of the complement of G , $\overline{\overline{G}}$, is isomorphic to G .

The complement of a graph has the same vertex set as the original graph, so the vertex set of $\overline{\overline{G}}$ must be the same as G . Now, the edge set of G is a subset of all pairs of vertices. By our definition, the complement of a graph has an edge set which is the complement with respect to all possible edges. Now, with arbitrary sets, we know that the complement of the complement is the original set. Therefore, $E(G) = E(\overline{\overline{G}})$. Since the edge sets and vertex sets are the same as G , we must have the same graph. Therefore, G is isomorphic to $\overline{\overline{G}}$.

6. Prove that the complement of a disconnected graph is connected.

We begin by assuming we have a disconnected graph G . Now consider two vertices x and y in the complement. If x and y are not adjacent in G , then they will be adjacent in \overline{G} and we can find a trivial x - y path. If x and y are adjacent in G then they must have been in the same component. Let z be some vertex in another component of G . This means that the edges xz and yz were not in G . This implies that they both must be edges in \overline{G} . This gives us the path x - z - y . Therefore, in \overline{G} we have that there exists a path between any two vertices and hence it is connected.

7. Let G be a graph which is isomorphic to its complement \overline{G} . Prove that G must have $4k$ or $4k + 1$ vertices for some integer k .

Since G is isomorphic to its complement, we know that both graphs have the same number of edges. Also, if we look at the union of the edges of both graphs we know we get all possible edges. If G has n vertices, there are $\frac{n(n-1)}{2}$

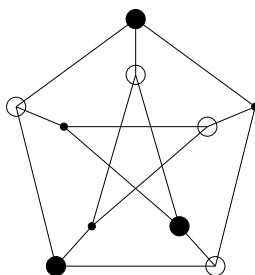
possible edges, so G must have exactly half of them. Therefore,

$$|E(G)| = |E(\overline{G})| = \frac{n(n-1)}{4}$$

Now we can see that since $|E(G)|$ must be an integer, 4 must divide evenly into n or $n-1$. So, we conclude that $n = 4k$ or $n = 4k+1$ for some non-negative integer k .

8. Find (with proof!) the chromatic number of the Petersen graph.

Since the Petersen Graph contains a 5-cycle as a subgraph, we know the chromatic number must be at least 3 (any odd cycle would do). Here is a particular colouring using 3 colours:



Therefore, we conclude that the chromatic number of the Petersen graph is 3.

9. Prove that if G is planar, then there must be some vertex with degree at most 5.

Assume for a contradiction that we have a planar graph where every vertex had degree at least 6. We know that for any graph, the sum of the degrees of the vertices always equals twice the number of edges. Therefore, if our graph has n vertices, we know that there must be at least $3n$ edges. We also know that for any planar graph, $e \leq 3n - 6$. This contradicts that we could have $3n$ edges. Thus, no such graph exists and we conclude that every planar graph has at least one vertex with degree at most 5.

10. Using the result of problem 9, prove by induction on $|V| = n$ that any planar graph can be coloured with at most 6 colours.

To begin, any graph with at most 6 vertices can be coloured with 6 colours. Now assume that we can colour any planar graph with k vertices with 6 colours. Now assume we have a graph with $k + 1$ vertices. We know there must be a vertex v with degree at most 5. We know the graph $G - v$ has k vertices, so by the induction hypothesis, it can be coloured with at most 6 colours. Now, since v only has 5 neighbours, we know that only 5 of the colours can be represented. Therefore, there must be some colour left over which we can assign to v . Now we've coloured G with at most 6 colours. So, by induction, any planar graph can be coloured with at most 6 colours.