CSCI/MATH 2113 January-April, 2005 Assignment 9, Answers.

1. Show whether each of the following sets under regular addition and multiplication is a ring.

(a) $S = \{a + \sqrt{3}b\}$ where a, b are in $\mathbb{R}$.

Answer: These are a subset of the real numbers, of a specific form, and therefore all the axioms hold as for real numbers. The additive inverse of $a + \sqrt{3}b$ is $-a - \sqrt{3}by$. All we have to check is closure under $+$ and $\cdot$. The sum of $a + \sqrt{3}b$ and $c + \sqrt{3}d$ is $(a + b) + (c + d)\sqrt{3}$, which is in the set.

The product

$$(a + \sqrt{3}b)(c + \sqrt{3}d) = (ac + 3bd) + \sqrt{3}(ad + bc),$$

and so S is closed under mutliplication.

(b) $S = \{a + \sqrt{3}b + \sqrt{5}c\}$ where a, b, c are in $\mathbb{R}$.

Answer: As in (a), the axioms are satisfied, but this time S is not closed under $\cdot$. For example, the product of $\sqrt{5}$ (which is $a = b = 0, c = 1$) and $\sqrt{3}$ (which is $a = c = 0, b = 1$) is $\sqrt{15}$, which is not in S. Therefore we do not have a ring.

2.

(a) Prove $(-a) \cdot b = -(a \cdot b)$. (Part of (f), page 6.)

Answer: From the other part of (f), page 6, we have $(c) \cdot (-d) = -(c \cdot d)$, since the equation is true $\forall a, b$. In this equation put $c = -a$ and $d = -b$. Then we get $(-a) \cdot (b) = -(-a \cdot -b) = - - (a \cdot b)$, by (e), page 6, which is $(a \cdot b)$ as required.

(b) Let $(S, \oplus, \circ)$ be a structure where $S = \mathbb{R}$, and $\oplus, \circ$ are defined by

$$\forall x, y \in S, \quad x \oplus y = x + y - 1; \quad x \circ y \quad = x + y - xy.$$

Is this structure a ring? Explain.

Answer: The structure is closed since the elements of S are real numbers and the result of each of the binary operators are also real. Hence we have to verify that the axioms hold.

(a) $\oplus$ is symmetric and so addition is commutative.

(b) Addition is associative since

$$x \oplus (y \oplus z) = x \oplus (y + z - 1) = x + y + z - 1 - 1.$$

and

$$(x \oplus y) \oplus z = (x + y - 1) \oplus z = x + y - 1 + z - 1 = x \oplus (y \oplus z).$$

(c) There is an additive identity, namely 1, since $x \oplus 1 = x + 1 - 1 = x$.

(d) There is an additive inverse for all elements in S. Since $x \oplus y = x + y - 1$, then $x \oplus y = 1$ if and only if $y = 1 + 1 - x$.

(e) $\circ$ is associative:

$$x\circ(y\circ z) = x\circ(y+z-yz) = x+y+z-yz-x(y+z-xy) = x+y+z-xy-xz-yz+xyz.$$

The final expression is symmetric in x, y, z and so we will get the same expression from$(x \circ y) \circ z$.

(f) The distributive rules hold. Here is the proof of

$$x \circ (y \oplus z) = (x \circ y) \oplus (x \circ z).$$

The other distributive law is proven similarly.

$$x \circ (y \oplus z) = x \circ (y + z - 1) = x + y + z - 1 - xy - xz + x.$$

$$x \circ y \oplus x \circ z = (x + y - xy) \oplus (x + z - xz) = x + y + z + x - xy - xz - 1.$$

Hence the distributive law holds.

3. Let $(S, +, \cdot)$ be a ring. Prove

(a) $a \cdot (b - c) = a \cdot b - (a \cdot c)$.

Answer: Since $a \cdot (b - c) = a \cdot b + a \cdot (-c)$ and $a \cdot (-c) = -(a \cdot c)$ (section 1.3, page 6, (f)), then $a \cdot (b - c) = a \cdot b - (a \cdot c)$.

(b) $(b - c) \cdot a = b \cdot a - (c \cdot a)$.

Answer: Here $(b - c) \cdot a = b \cdot a + (-c) \cdot a = b \cdot a - c \cdot a$, using question 2, above.

4. Let $(S, +, \cdot)$ be a ring.

(a) Prove that a unit in the ring cannot also be a divisor of zero.

Answer: If a is a unit, then a^{-1} exists. Now if a is a divisor of zero it follows that

$$\exists b \quad a \cdot b = z, \quad a \neq z, b \neq z.$$

But then $z = a^{-1} \cdot z = a^{-1} \cdot a \cdot b = b$, and so $b = z$, which is a contradiction. Therefore a cannot be a divisor of zero.

(b) If $a, b \in S$ are units, is $(a + b)$ a unit? Prove your claim.

Answer: No, $a + b$ need not be a unit. In the ring $\mathbb{Z}_6$, 1 and 5 are both units ($5^{-1} = 5$), but (1+5) is 0, not a unit.

In general, if a ring has a unity, u, then both u and $(-u)$ are units, and $u + (-u) = z$.

5. Below are tables for a ring with elements $\{s, t, x, y\}$. Using the axioms for a ring, fill in the missing entries in the multiplication table.

Is this a commutative ring? Does it have a unity? Are there any units? Is the ring an integral domain or a field? Prove your claims.

$+$	s	t	x	y
s	s	t	x	y
t	t	s	y	x
x	x	y	s	t
y	y	x	t	s

$\cdot$	s	t	x	y
s	s	s	s	s
t	s	t	?	?
x	s	t	?	y
y	s	?	s	?

Answer:

$+$	s	t	x	y
s	s	t	x	y
t	t	s	y	x
x	x	y	s	t
y	y	x	t	s

$\cdot$	s	t	x	y
s	s	s	s	s
t	s	t	x	y
x	s	t	x	y
y	s	s	s	s

First compute, $x \cdot x$.

$x \cdot (x + t) = x \cdot y = y$ according to the addition table.
Therefore $x \cdot x + x \cdot t = y = x \cdot x + t$ according to the multiplication table.
Therefore $x \cdot x + t = y$, and so $x \cdot x = y + (-t) = y + t = x$.

Now, compute $y \cdot t$.
$(y + x) \cdot t = t \cdot t = t$.
Therefore $y \cdot t + x \cdot t = t$. Hence $y \cdot t + t = t$. Therefore $y \cdot t = s$.

Compute $y \cdot y$.
$y \cdot (t + x) = y \cdot y \cdot y$. Also:
$y \cdot (t + x) = y \cdot t + y \cdot x = y \cdot t + s$.
Hence $y \cdot y = y \cdot t = s$.

Compute $t \cdot x$.
$(y + t) \cdot x = x \cdot x = x$. Also:
$(y + t) \cdot x = y \cdot x + t \cdot x = s + t \cdot x$. Therefore
$t \cdot x + s = x$. That is, $t \cdot x = x$.

Finally, compute $t \cdot y$.
$(x + t) \cdot y = y \cdot y = s$. And: $(x + t) \cdot y = x \cdot y + t \cdot y$. Therefore
$x \cdot y + t \cdot y = s$. That is, $x \cdot y = -t \cdot y$. Therefore $t \cdot y = -y = y$.

6. Let $\alpha = 2^{\frac{1}{3}}$. Prove that

$$(\{a + b\alpha + c\alpha^2\}, +, \cdot)$$

for $a, b, c \in \mathbb{R}$, is a ring under the usual addition and multiplication.

Answer: These are simply a subset of the real numbers. The zero is $a = b = c = 0$. Therefore the axioms hold. There is also a unity, 1. We only have to verify closure.

The sum of $a + b\alpha + c\alpha^2$ and $d + e\alpha + f\alpha^2$ is $(a + d) + (b + e)\alpha + (c + f)\alpha^2$, which is also in the set. Therefore we have closure under $+$.

The product of $a + b\alpha + c\alpha^2$ and $d + e\alpha + f\alpha^2$, using the fact that $\alpha^3 = 3$, is

$$(ad + 3bf + 3ce) + (ae + bd + 3cf)\alpha + (af + cd + be)\alpha^2,$$

which is still in the set. Therefore this is a ring.

7. Let $R = (S, +, \cdot)$ be a ring which is not a field. Is it possible for R to have a subring which is a field? Either prove that this is not possible or give an example of a ring with divisors of zero which has a subring which is a field.

Answer: Take $\mathbb{Z}_6$. This is not a field since $2 \cdot 3 = 0 \bmod 6$. But within the ring, there is a subring $\{0, 2, 4\}$ which is a field mod 6. The unity is 4.

Note: If a ring has a unity and divisors of zero, then any subring which includes the unity will have divisors of zero. (*Why?*).

8. Let D be an integral domain. Prove that if $a^2 = 1$ then $a = \pm 1$. Is this true in a ring also? Prove it, or give a counter example.

Answer: If $a^2 = 1$ then $a^2 - 1 = z$, and therefore $(a-1)\cdot(a+1) = z$. Hence, since D is an integral domain (with no divisors of zero) we must have $a = 1$ or $a = -1$.

This is not true in a general ring. Take, for example, $\mathbb{Z}_8$, with $a = 3$. Then $a^2 = 9 = 1 \bmod 8$. But $(a-1)(a+1) = 2 \cdot 4 = 0 \bmod 8$, and $a \neq 1, a \neq -1$.

9. Exercise 2, Section 1.6, page 8 of the notes. (Note the definitions in exercise 1.)

Answer: (a) Let the ring be finite, size k. Then form $e, e+e, e+e+e, \cdots ke$. These are all distinct, for if $je = ie, i < j \leq k$ then $(j-i)e = z$, and so the characteristic is finite. If they are distinct then one of these k elements must be z, i.e. $je = z$ for some $j \leq k$. That is, the characteristic is finite.

(b) Using the same arguments as in (a), if a is in the finite ring, and the size of the ring is k, then there is some $j \leq k$ such that $je = a$. Then $na = nje = (nj)e = (jn)e = j(ne = z$, where we are using the properties of integers.

(c) Take $\mathbb{Z}_6$, and look at the element 2. The characteristic is 6. But $3 \cdot 2 = 0$ in this ring.

10. Let S be a set, and $P(S)$ the power set of S. Prove that $(P(S), \Delta, \cap)$ is a ring, where Δ is the symmetric difference, and $\cap$ is set intersection. Is this a ring? Prove your claim.

Answer: We have to show that the axioms are satisfied. The set is clearly closed under Δ and $\cap$.

(a) $A \Delta B = (A \cup B) - (A \cap B) = (B \cup A) - (B \cap A) = B \Delta A$, by the properties of $\cap$ and $\cup$.
(b) The associative axiom holds because of the set theoretic properties of Δ.
(c) The additive identity is ϕ since $A \Delta \phi = A$.

(d) The additive inverse of A is A itself. Recall the example in class where the "plus" of the ring was $\cup$. It was not possible to define an additive identity. But here $A \Delta A = \phi$.
(e) $A \cap (B \cap C) = (A \cap B) \cap C$ is a property of $\cap$.
(f) $A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$ was an assignment example in csci 2112. I will do this in a tutorial if necessary.