MATH 3790 - Assignment 1

Due: Sept 23

September 17, 2003

1. Prove that $\sqrt[3]{3}$ is irrational.

   First assume for a contradiction that $\sqrt[3]{3}$ is a rational number. Then we can write $\frac{p}{q} = \sqrt[3]{3}$. Rearranging, we get

   $$p^3 = 3q^3$$

   We can now note that the number of 3's in the prime factorization of the left side is a multiple of 3. On the right side however, there is one more than a multiple of three. Therefore, by the unique factorization theorem, we know that these numbers are not the same and we get a contradiction. Thus $\sqrt[3]{3}$ is irrational.

2. Prove that $\displaystyle\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2$.

   From class we know that $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$. Therefore, we simply need to show that $\displaystyle\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$. We will proceed by induction.

   For a base case, we choose $n = 1$ and see that $\displaystyle\sum_{i=1}^{1} i^3 = 1$ and $\left(\frac{1(1+1)}{2}\right)^2 = 1$. Therefore, the statement is true for $n = 1$.

   Now we assume that the statement is true for $n = k$. That is, $\displaystyle\sum_{i=1}^{k} i^3 =$

$\left(\frac{k(k+1)}{2}\right)^2$. Now we look at the statement for $n = k+1$:

$$\begin{aligned}
\sum_{i=1}^{k+1} i^3 &= \sum_{i=1}^{k} i^3 + (k+1)^3 \\
&= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 \\
&= \frac{k^4+2k^3+k^2}{4} + k^3 + 3k^2 + 3k + 1 \\
&= \frac{k^4+2k^3+k^2}{4} + \frac{4k^3+12k^2+12k+4}{4} \\
&= \frac{k^4+6k^3+13k^2+12k+4}{4} \\
&= \frac{(k+1)^2(k+2)^2}{4} \\
&= \left(\frac{(k+1)(k+2)}{2}\right)^2
\end{aligned}$$

Therefore, by induction we conclude that the statement is true for all n.

3. Given exactly one of each of the Tetris pieces (there are 5 of them), determine if they can be arranged so that they form a rectangle without any overlapping.

   If we examine all the pieces when placed on a grid coloured like a standard checkerboard we find that each piece covers exactly two white and two black squares except the piece that is in the shape of a T which covers a single square of one colour and three of the other. When all the pieces are placed together, we know they will cover and odd number of white squares. Any rectangle coloured in such a way will always have an even number of white squares so there is no way to arrange them so that every square is covered without overlap.

4. An ancient puzzle called the *Tower of Hanoi* consists of three pegs on a stand and n punctured discs of different sizes that are placed in decreasing order on one of the pegs. The object of the puzzle is to

transfer the pile of discs to another peg, by moving one disc at a time, and without placing any disc on top of a smaller disc. Show that it is possible to solve this puzzle in $2^n - 1$ moves.

We will proceed by induction on the number of discs. For the base case, we consider the puzzle when there is only 1 disc. We can move this disc to the other peg in one move which finishes the puzzle. Since $1 = 2^1 - 1$ the statement is true in this case.

Now assume that we can solve the puzzle with k discs in $2^k - 1$ moves. We now examine the puzzle when there are $k+1$ discs. If the discs are on peg A and we want to move them to peg C, we know by induction that we can move the first k discs to peg B in $2^k - 1$ moves. Then we can move the bottom disc to peg C using 1 move and finally moving the k discs from peg B to peg C using another $2^k - 1$ moves. In total we have used

$$2^k - 1 + 1 + 2^k - 1 = 2 \cdot 2^k - 1 = 2^{k+1} - 1$$

moves. Therefore, by induction we can solve the puzzle with n discs in $2^n - 1$ moves for all n.

5. Do there exists non-zero integers a and b such that one of them is divisible by their sum and the other is divisible by their difference?

   First we assume that there are non-zero integers a and b such that $\frac{a}{a+b} = n$ and $\frac{b}{a-b} = m$ for some integers n and m. Then, we can solve for a in terms of n and b:

$$\begin{aligned} \frac{a}{a+b} &= n \\ a &= an + bn \\ (1-n)a &= bn \\ a &= \left(\frac{n}{1-n} b\right) \end{aligned}$$

   Now, we substitute into the other equation to solve for m in terms of n:

$$m = \frac{b}{a-b}$$

$$\begin{aligned} &= \frac{b}{\left(\frac{n}{1-n}\right)b - b} \\ &= \frac{b}{\left(\frac{2n-1}{1-n}\right)b} \\ &= \left(\frac{1-n}{2n-1}\right) \end{aligned}$$

Without loss of generality we can assume that n is non-negative. If $n = 0$ then we get that $a = 0$ which is not allowed. Similarly, if $n = 1$ then we get that $b = 0$ which is again not allowed. For $n \geq 2$ we find that $|2n - 1| > |1 - n|$ so m will never be an integer. This is a contradiction. Therefore, there are no pair of numbers a, b with the given property.

6. (BONUS) Prove that e is irrational.

   As a hint - try writing that

$$\frac{p}{q} = e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots$$

   then multiplying both sides by q!