MATH 3790 - Assignment 2 Solutions

Due Oct 16

October 17, 2003

1. Show that there is no polynomial p with integer coefficients such that $p(1) = 4$ and $p(3) = 7$.

We begin by assuming that such a polynomial exists, and examine the following equations and taking their difference:

$$\begin{aligned} p(3) &= a_0 + a_1(3) + a_2(3)^2 + ... = 7 \\ p(1) &= a_0 + a_1(1) + a_2(1)^2 + ... = 4 \\ p(3) - p(1) &= a_0(1-1) + a_1(3-1) + a_2(3^2 - 1^2) + ... = 7 - 4 = 3 \end{aligned}$$

Now we see that every term in the last equation is an even number since each coefficient is multiplied by the difference of two odd numbers. But then we see that we have a sum of only even numbers is 3. This is a contradiction, so such a polynomial cannot exist.

2. Prove that the sequence $\{11, 111, 1111, 11111, ...\}$ contains no perfect squares.

First, we consider all of the elements of the sequence modulo 4. Each of the terms can be written as $100n + 11$ for some integer n. Then we note that $100n + 11 \equiv 11 \equiv 3 \pmod 4$. From class we know that the quadratic residues modulo 4 are $\{0, 1\}$. Since every element of the sequence is congruent to 3, none are perfect squares.

3. In the magical land of Camelot, there are 45 chameleons. Thirteen are lavender, fifteen are beige and seventeen are aquamarine. Whenever two chameleons of different colours meet, they both change into a chameleon of

the third colour. Prove that no matter how many times these chameleons meet, it is impossible for all of the chameleons to be of the same colour at one time.

To begin, we will consider all the values modulo 3. We begin with $(L, B, A) = (13, 15, 17) \equiv (2, 0, 1) \pmod 3$. Now consider what happens to the numbers of chameleons when they meet. Two sets go down by 1 and the other goes up by 2. When we deal with numbers modulo 3, we know that $-1 \equiv 2$ so the act of meeting affects all groups in the same way. After one meeting (regardless of which two meet each other!) we will have $(L, B, A) \equiv (1, 2, 0) \pmod 3$. After the second meeting we get $(L, B, A) \equiv (0, 1, 2) \pmod 3$ and after the third we return to $(L, B, A) \equiv (2, 0, 1) \pmod 3$. Since there is no other way for the chameleons to interact, we know that these are the only 3 triples (modulo 3) that can ever occur regardless of the order of their interactions. In particular, we can never get the scenario where there are 45 chameleons of the same colour since there would be 0 of each of the other colours giving $(L, B, A) = (0, 0, 0) \pmod 3$. Therefore, it can never be the case that all the chameleons are the same colour.

4. Pick any 5 lattice points in the plane (points where the x and y coordinates are both integers). Prove that the midpoint of a line segment defined by some pair of these points is again a lattice point.

We begin by using the pigeonhole principle. The pigeons are the five lattice points which are selected. The pigeonholes will be the four types of points based on the parity of each coordinate. Therefore, we have (even, even), (even, odd), (odd, even) and (odd, odd). By the pigeonhole principle, we know that there are two lattice points that have the same parity in both coordinates. Now, the midpoint between points (x_1, y_1) and (x_2, y_2) is given by $(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})$. Since we know the parity of the coordinates are the same, we know that both $x_1 + x_2$ and $y_1 + y_2$ are even numbers and hence divisible by 2. Therefore, the values $\frac{x_1+x_2}{2}$ and $\frac{y_1+y_2}{2}$ integers and hence $(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})$ is a lattice point.

5. Show that there is an integer whose digits are all 0s and 1s which is divisible by 2003.

We will proceed by using the pigeonhole principle. We will let the pigeonholes be the congruence classes modulo 2003. We will let the pigeons be the numbers of the form 111...11 where there are between 2 and 2005 1s. Since there are 2004 such numbers, by the pigeonhole principle we know that two of them are in the same congruence class modulo 2003. Let these two numbers be a and b (and without loss of generality assume $a > b$). Then we know that $a \equiv b \pmod{2003}$. which implies that $a - b \equiv 0 \pmod{2003}$. Therefore, $a - b$ is divisible by 2003. Also, since both a and b are composed entirely of 1s, then their difference is composed of only 1s and 0s. Therefore, there is a number (namely $a-b$ composed of only 1s and 0s that is divisible by 2003).

6. (BONUS) Find all integer solutions (x, y, z) to the equation $x^2 + y^2 = 3z^2$.

Hint: Consider the equation modulo 3. What does this tell us about x and y? Then assume that there is a *smallest* non-trivial solution and find a contradiction. (Note: This is similar to the geometric proof we did with $\sqrt{2}$ being irrational.)