MATH 3790 - Assignment 3 Solutions

Due Nov 6

November 6, 2003

1. State and prove the AM-GM inequality for 3 terms.

We need to show that for three non-negative integers x, y, z we have that $\frac{x+y+z}{3} \geq \sqrt[3]{xyz}$. To begin, we make the substitution $x = a^3$, $y = b^3$, $z = c^3$. Now we need to show that $a^3 + b^3 + c^3 \geq 3abc$.

From class we have shown that $a^2 + b^2 + c^2 - ab - ac - bc \geq 0$. Also, we know that $a + b + c \geq 0$ since $a, b, c \geq 0$. Now, we can factor as follows:

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - ac - bc) \geq 0$$

Rearranging the first and last terms we get the desired result.

2. Consider the sequence $\{1, \sqrt{2}, \sqrt[3]{3}, \sqrt[4]{4}, ...\}$. Which term is the largest? Prove your claim.

I claim that $\sqrt[3]{3}$ is the largest value in the sequence. It is easy to check that $\sqrt[3]{3} > \sqrt{2} > 1$ so we must now show that $\sqrt[3]{3} \geq \sqrt[k]{k}$ for all $k \geq 3$. Raising both sides to the power of $3k$ we eliminate the roots and now require that $3^k \geq k^3$. One way to show this is true is by induction. The base case will be when $k = 3$ where the inequality holds exactly. Now we assume it holds for $n = k$ and try to prove the case for $n = k + 1$.

$$\begin{aligned} 3^{k+1} &= 3^k + 3^k + 3^k \\ &\geq k^3 + 3^k + 3^k \\ &> k^3 + 3k^2 + 3k + 1 \\ &= (k+1)^3 \end{aligned}$$

the strict inequality can be seen when you compare the terms $3k^2$ and $3k+1$ to the terms 3^k from the previous line. This can be made more formal but is not required.

3. Find all real x, y, z such that

$$-2 < \frac{x^2 + y^2 + z^2}{xy + xz + yz} < 1$$

If x, y, z are all positive or all negative then the denominator is positive so we can multiply on the right to get the inequality $x^2 + y^2 + z^2 < xy + xz + yz$ which we've seen in class can be rearranged and factored to give $(x-y)^2 + (x-z)^2 + (y-z)^2 < 0$. Of course, this cannot have any solutions. Therefore, not all of x, y, z can be positive or negative. Also, we note that if y and z are both negative, both the numerator and denominator have the same value as if we change all their signs. By symmetry, we can assume without loss of generality that x is negative and y and z are positive. So, we replace x by $x' = -x$ so that all the terms are positive. Then, multiplying on the left side we get the equation $(x')^2 + y^2 + z^2 < 2x'y + 2x'z + 2yz$. We can further simplify this to the inequation $(x')^2 + (y+z)^2 < 2x'(y+z)$.
Since x', y, z are all positive we know by the AM-GM that $\frac{(x')^2+(y+z)^2}{2} \geq x'(y+z)$ which contradicts our previous statement. Therefore there are no real solutions to the given inequality.

4.a) How many solutions are there to the equation $a_0 + a_1 + a_2 = 12$ for integers $a_i \geq 1$?
b) How many solutions are there to the equation $a_0 + a_1 + a_2 = 9$ for integers $a_i \geq 0$?

a) First, we note that we have 12 'objects' which we want to split into the three variables a_0, a_1, a_2. To view the problem visually, draw 12 xs in a row and draw 2 lines splitting them into 3 parts. So, each solution is equivalent to choosing 2 lines from the available 11 positions. Therefore, there are $\binom{11}{2} = 55$ different solutions.

b) Instead of solving a whole new problem, we can just give a bijection between solutions to $a_0 + a_1 + a_2 = 12$ and solutions so $b_0 + b_1 + b_2 = 9$. In this case, we let $b_i = a_i - 1$. Since $a_i \geq 1$ we now have that $b_i \geq 0$ and

$$\begin{aligned} a_0 + a_1 + a_2 &= 12 \\ a_0 + a_1 + a_2 - 3 &= 12 - 3 \\ a_0 - 1 + a_1 - 1 + a_2 - 1 &= 9 \\ b_0 + b_1 + b_2 &= 9 \end{aligned}$$

Therefore we also have 55 solutions to this problem as well.

5.a) Prove that the sum of the elements in the n^{th} row of Pascal's triangle is 2^n.
b) Prove that the alternating sum of the elements in the n^{th} row of Pascal's triangle is 0.

a) There are at least 3 different solutions to this problem. I give the shortest one here. From the binomial theorem we know that

$$(1+1)^n = \sum_{i=0}^{n} 1^i \binom{n}{i}$$

The left side is 2^n while the right side is the sum of the elements in the n^{th} row of Pascal's Triangle.

b) As before we use the binomial theorem to get:

$$(1-1)^n = \sum_{i=0}^{n} (-1)^i \binom{n}{i}$$

In this case the left side is 0 and the right side is the alternating sum of the n^{th} row of Pascal's Triangle.