MATH 3790 - Assignment 4

Due Nov 27

November 20, 2003

1. Given a graph G with n vertices, prove that if $d(v) \geq \frac{n}{2}$ for all $v \in V(G)$ then G is connected.

Assume for a contradiction that G is not connected. Then there are 2 vertices x, y which are in different components. Since x has degree at least $\frac{n}{2}$, the component containing x must have at least $\frac{n}{2} + 1$ vertices. The same can be said for the component containing y. But then we have the total vertices in both components is at least $n + 2$ which contradicts the fact $|V(G)| = n$.

2. Prove that K_5 is not planar.

As seen in class, we know that for a planar graph with p vertices and q edges we get that $q \leq 3p - 6$. K_5 has 5 vertices and 10 edges. Since we can see that $10 > 3(5) - 6 = 9$ we know that K_5 has too many edges to be planar. Alternatively, this can be proven using the pigeonhole principle together with the Jordan Curve Theorem in a manner much like the proof that $K_{3,3}$ is not planar done in class.

3. A graph G is said to have an *Euler walk* from u to v if there is a walk from u to v which uses every edge of G exactly once. Characterize all graphs which contain an Euler walk.

Claim: A graph G has an Euler walk if and only if it is connected and has exactly 2 vertices have odd degree.

Proof: First, assume we have a graph with two odd-degree vertices u and v. Construct an Euler walk from u to v by first taking any walk W from u to v.

If this is an Euler walk, we are done, otherwise, there are vertices which have incident edges which have not been included in the walk - let x be such a vertex (note that x could be the same as either u or v). Construct a walk A that begins and ends at x that doesn't use any of the edges occurring in W. This can always be done because every vertex has an even number of edges incident with them which are not part of W (the idea here is that when you go to a vertex along an edge, there is always another edge by which you can leave that vertex unless you've arrived back at x). We now augment W taking the walk from u to x, then traversing A then taking the walk from x to v. Repeat this process until all edges of G are included in the walk.

To show all other graphs do not have an Euler walk, we first note that if all vertices have even degree, by a simple parity argument we see that after traversing all the edges incident to the vertex we started at (say u), we must be on that vertex and hence cannot get to v to create a u-v walk. Therefore, the beginning and ending of an Euler walk must be vertices of odd degree. If there are more than 2 vertices with odd degree, then consider a vertex x with odd degree. In order to traverse all edges incident with x the same parity argument as above shows us that we will end on x (since the first traversal was *to* x). In this case, it will be impossible to find an Euler walk.

Therefore the original claim is proven.

4. The game of Clobber is played on a 6x7 checkerboard with alternating black and white pieces. On each player's turn they move a piece of their colour one space horizontally or vertically onto a space which must contain a piece of the opposite colour. This piece is 'clobbered' and removed from the board (being replaced by the one that moved). When a player cannot make a legal move, they lose. Find out as much as possible about this game. (This problem will be worth twice that of the others)

There are many things that can be said about this game. Points are generally awarded for 'intuitive' strategies for playing as well as determining the winner of classes of simple positions.

5. (Bonus) Given a graph G with n vertices, what is the maximum number of edges G can have if there are no three vertices which are mutually adjacent (ie no triangles)?