ANALYSIS OF THREE NEW COMBINATORIAL GAMES

By

Paul Ottaway

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DEPARTMENT OF
MATHEMATICS AND STATISTICS

The undersigned hereby certify that they have read and recommend to the Faculty of Graduate Studies for acceptance a thesis entitled "**Analysis of three new combinatorial games**" by **Paul Ottaway** in partial fulfillment of the requirements for the degree of **Master of Science**.

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Supervisor: ______________________
Richard J. Nowakowski

Readers: ______________________
Jeanette Jansen

____
Jason Brown

DALHOUSIE UNIVERSITY

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Author: **Paul Ottaway**

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Abstract

This thesis will present three combinatorial games: Vertex Deletion, Grand Left/Right and Cookie Cutter. Vertex Deletion is a game played on a graph. One version of the game is completely solved while the values of games played on complete graphs, paths, cycles, complete bipartite graphs and stars are examined in others. For a particular version we also demonstrate a decomposition theorem. We introduce the notion of even and odd games and show that it relates to both Vertex Deletion and Grand Left/Right. Cookie cutter is played by removing squares from a grid of a fixed size. We find a relation between this game and a set of octal games. Starting positions with one and two rows are completely solved in one version of the game.

Acknowledgements

I would like to thank Richard Nowakowski for all his help and support over the last year. He has made mathematics even more of a joy than I ever could have expected. I would also like to thank my family and friends for always believing in me and giving me the strength to achieve my goals.

Chapter 1

Theory of Combinatorial Games

1.1 Properties

All three of the games which I will examine are combinatorial games. There are a specific set of conditions which a game must satisfy in order to qualify as combinatorial. They are as follows:

- There are two players, typically denoted **Left** and **Right**, who move alternately during the course of the game. By convention, Left is male and Right is female.
- There are clearly defined rules which specify the moves allowed to each player.
- There is complete information. That is, all information regarding the game is equally available to both players.
- There is no element of chance. Therefore, there is never any use of dice, spinners or similar devices.
- Under **normal play** rules, a player who is unable to make a legal move loses the game. Under **misere play** rules, a player who cannot move wins the game. Unless otherwise stated, all games discussed will use normal play.
- There can be at most a finite number of moves allowed in the game. Therefore, the game must eventually end with one winner and one loser.

Many common games do not qualify as combinatorial since they violate one or more of these properties. For example, card games such as poker have cards randomly (hopefully!) assigned. Also, players are only allowed to view they're own cards and not those of their opponents. Therefore, there is incomplete information. Games like chess have no element of chance and complete information but still do not qualify as combinatorial since the game can end in a draw (stalemate) when a player has no legal moves or when neither player can force a checkmate. The tools of combinatorial game theory have been used, however, to analyze such games although some modifications need to be made. The analysis of games like chess, checkers and go has been furthered using combinatorial game theory.

The game of **Nim** is played by two players with several piles of beans. On each player's turn, they choose any one of the remaining piles then remove any positive number of beans from that pile. The person who is unable to do so loses the game. It is easy to see that this is indeed a combinatorial game since it satisfies all of the properties listed above. Furthermore, Nim is called an *impartial* game since from any particular position either player has the same options.

Partizan games on the other hand allow players to have a different set of options from a particular position. **Domineering** is an example of such a game. It is played by two players with dominoes and a square grid, typically 8×8. The players alternate placing dominoes on the board so that they cover exactly two adjacent squares without overlapping the edge of the board or dominoes which have already been placed. When a player cannot place a domino according to these rules, they lose. Left may only place his dominoes vertically while Right may only place hers horizontally. Given a particular position during the game, the players will have a different set of options thus making this a partizan game.

The rest of this chapter aims to describe the theory of combinatorial games. The theorems presented are taken from [2] and [6]. Some of the proofs are the result of a

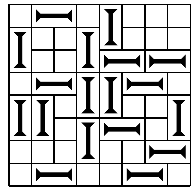


Figure 1.1: A typical game of Domineering in progress

seminar on combinatorial game theory held at Dalhousie University which included Richard J. Nowakowski, J.P. Grossman, Sarah McCurdy and myself (see also [8]).

1.2 Outcome Classes

The outcome of a combinatorial game is completely dependant on the moves which are allowed to each player and which player has the privilege (or misfortune) of moving first. Since combinatorial games have no chance and complete information it is always possible to determine who will win the game before any moves are made. The only assumption we must make is that both players play as well as possible. Therefore, we can partition all games into the following four classes:

L - All games which the Left player will win regardless of who moves first.

R - All games which the Right player will win regardless of who moves first.

P - All games which the second (previous) player will win regardless of whether they are Right or Left.

N - All games which the first (next) player will win regardless of whether they are Right or Left.

Clearly, every game must fall into one of these classes and we can also see that the classes are all disjoint. Here are some examples of positions in Domineering which are from each class:

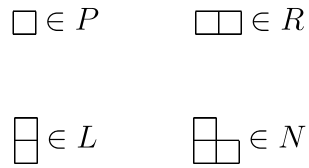


Figure 1.2: Domineering positions from each outcome class

1.3 Game Values

Games can be defined recursively in terms of the options that each player has from a given position. Let G be a game. It is best to think about G as a given game in a particular position but without knowledge of which player is to play next. Then $G = \{G^L|G^R\}$ where G^L is the set of all possible games (positions) that can be reached if Left moves first. Likewise, G^R is the set of all possible games that can be reached if Right moves first. In this way we recursively define games based on their Left and Right options. It should be noted that notation is occasionally abused and we use the symbols G^L and G^R to mean a specific option rather than the set of all options for the given player.

Clearly our most basic game in this sense is one in which neither player has any moves. Then this game has G^L and G^R both being the empty set. This game is represented by the symbol $0 = \{\,|\,\}$. We may now form three new games:

$$\{0|\,\}, \{\,|\,0\}, \{0|\,0\}$$

We give these games the symbols 1, -1 and $\star$ respectively. We will show later that there is motivation to name these games using numerical symbols since they share all the properties of their more familiar counterparts. Following Conway [6], I will now give definitions for comparing games:

For games G and H,

- $G \geq H$ if and only if no $G^R \leq H$ and $G \leq$ no H^L. Also, we write $G \not\geq H$ when $G \geq H$ is false.
- $G \leq H$ iff $H \geq G$.

- $G = H$ iff $G \geq H$ and $H \geq G$.

- $G > H$ iff $G \geq H$ and $H \not\geq G$. Likewise, $G < H$ iff $H > G$.

- $G||H$ iff $G \not\geq H$ and $H \not\geq G$. We say G is **fuzzy** (or **confused**) with H in this case.

- $G \lhd H$ iff $G < H$ or $G||H$. Likewise, $G \rhd H$ iff $H \lhd G$.

It is important to note that the symbols $\not\geq$ and $<$ cannot be used interchangeably because of the notion of two games being confused. Also, equality only refers to the value assigned to a particular game. It is possible to have games with different options that are equal according to the definitions above.

The definition of $\geq$ is recursive in nature since it depends on the comparisons of the game's options. Since all games must eventually come to an end, we are reduced to asking questions about members of the empty set which are trivial. Therefore, $\geq$ is well defined.

We can now examine the games we have labelled so far to check that they satisfy the properties shared by their numerical counterparts. For instance, we can check that $1 > 0$ by using the previous definitions and the games that we have labelled 1 and 0.

First we note that $1 > 0$ is the same as saying $1 \geq 0$ and $0 \not\geq 1$. We check that $1 \geq 0$ by recalling that $1 = \{0|\,\}$ and $0 = \{\,|\,\}$. 1^R and 0^L are both the empty set, so it is trivial that no $1^R \leq 0$ and $1 \leq$ no 0^L. Therefore, $1 \geq 0$ by our definition.

Now, to check $0 \not\geq 1$ we note that $1^L = 0$. In particular, that means there exists an 1^L which satisfies $0 \leq 1^L$. We can therefore conclude that $0 \not\geq 1$.

These two properties imply that $1 > 0$. Using similar arguments we can show all

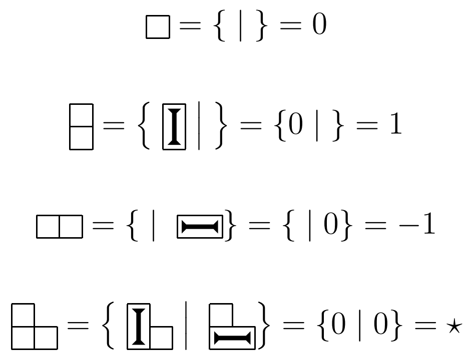


Figure 1.3: Finding values of Domineering positions

of the properties that we are familiar with using -1, 0 and 1. We also get the following comparisons with the game labelled $\star$:

$$\star > -1 \;\;,\quad \star < 1 \;\;,\quad \star || 0$$

Definition 1.3.1 *A game G is **born on day n** if it can be expressed as a game with options which are all born on day $n-1$ or less and there is no expression for G with options which are all born on day $n-2$ or less. We define 0 to be born on day 0. We say the **birthday** of G is n.*

Theorem 1.3.1 *[6] For all games $G = \left\{ G^L \;\middle|\; G^R \right\}$ we have*

1. $G \not\geq G^R$
2. $G^L \not\geq G$
3. $G \geq G$
4. $G = G$

Proof: We will prove all four statements using induction based on the day a game is born. We can show that $0 \leq 0$ by applying the above definitions. We will now proceed by assuming that $H \leq H$ for all games born before G.

1. Since G^R is, by definition, born before G we know that $G^R \leq G^R$ and so it follows that $G \not\geq G^R$.

2. Since G^L is born before G we know that $G^L \geq G^L$ and so $G \ngeq G^R$.

3. This follows immediately by applying the first two statements.

4. By the previous statement we know $G \geq G$ and $G \leq G$ so by definition $G = G$.

■

1.4 Outcomes Revisited

Now that we have the values of games, we can see how these values fit in with our notion of outcome classes.

Theorem 1.4.1 *For all games G, outcome classes are equivalent to sets of games that satisfy the following relations:*

- $G \in L \Leftrightarrow G > 0$
- $G \in R \Leftrightarrow G < 0$
- $G \in P \Leftrightarrow G = 0$
- $G \in N \Leftrightarrow G || 0$

Proof: We will prove the result by induction on the birthday of G. We have already seen that the above properties hold for the games 0, -1, 1 and $\star$ which are all born by day 1. This will serve as our base case. Now assume the above relations hold for all games born by day k. We now use our previous definitions to examine the options each player has in a general game G.

- If $G > 0$ then we know that all of Right's options are > 0 and Left's must have some option ≥ 0. Thus, Right loses going first since he must move to a game in L. Also, Left can win going first since he can move to a game in $L \cup P$. Therefore, $G \in L$.

- If $G < 0$, we use a similar argument to show Right has an option to a game in $R \cup P$, while Left only has options in R. Thus we have that $G \in R$.

	Left plays first, Left wins	Left plays first, Right wins
Right plays first, Left wins	$G > 0$ $G \in L$	$G = 0$ $G \in P$
Right plays first, Right wins	$G \vert\vert 0$ $G \in N$	$G < 0$ $G \in R$

Table 1.1: Summary of Outcome Classes

- If $G = 0$ then we know that all of Left's options are $\lhd 0$ which means he would lose going first. Also, all of Right's options are $\rhd 0$ which means she also loses if required to play first. Therefore, the second player will always win and $G \in P$.
- Finally, if $G||0$ we know that there exists a Left option which is ≥ 0 which means he can win if he plays first. Likewise, Right must have an option which is ≤ 0 which would allow her to win if she played first. Therefore, $G \in N$.

■

1.5 Negatives, Sums and Comparisons of Games

I will now define some further operations on games. The negative of a game can be thought of intuitively as Right and Left switching roles. Formally, for a game $G = \{G^L|G^R\}$ we define its negative as $-G = \{-G^R| - G^L\}$.

In the game of Domineering, since one player only places dominoes vertically and the other only places them horizontally we can view the negative of the game as being the same remaining spaces rotated ninety degrees. For example, consider the figure below.

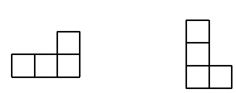


Figure 1.4: A game of Domineering and it's negative

The sum of two games can be thought of as both games sitting side by side. On a player's turn they must make a move in one of the two games. Formally, this becomes:

$$G + H = \{G^L + H, G + H^L | G^R + H, G + H^R\}$$

Definition 1.5.1 *The **disjunctive sum** of games $G_1, G_2, ...$ is $G_1 + G_2 + ...$ On each player's turn, they choose a game to play in and make a legal move. When a player has no legal move in any game, he loses.*

Theorem 1.5.1 *For any game G, $G + (-G) = 0$.*

Proof: Intuitively, this is the same as playing two identical copies of a game with the roles of the players reversed for one of them. The second player will win this compound game, since he can simply copy his opponent's move in the opposite game every time it is his turn. Therefore, he cannot possibly be left without a move and thus eventually wins the game. Since the second player always has a winning strategy, this game has value 0. Therefore, we arrive at the identity $G + (-G) = 0$. ∎

Theorem 1.5.2 *For any two games G and H, $(G = H) \Leftrightarrow G + (-H) = 0$.*

Proof: We will again prove the statement by induction on the birthdays of G and H. By our definition of $>$, we know that all $G^L \lhd H$ and all $G^R \rhd H$. Also, we get that all $H^L \lhd G$ and all $H^R \rhd G$.

Now, when we consider the game $G - H$ we see that Left's moves are either to $G^L - H \lhd H - H = 0$ or $G - H^R \lhd G - G = 0$. In either case we find that Right will win the game so $G - H \in R \cup P$. Likewise, if Right plays first her options are to $G^R - H \rhd H - H = 0$ or $G - H^L \rhd G - G = 0$. This time we find that Left will win which tells us $G - H \in L \cup P$. Therefore, $G - H \in P$ which in turn tells us that $G - H = 0$. ∎

This now gives us a method for determining if two games are equal. For any two games G and H, if it is true that $G + (-H) = 0$ then we know that $G = H$. Another way of saying this is that if G is played with $-H$ and the game is a second player

$G + H$

$G + (-H)$

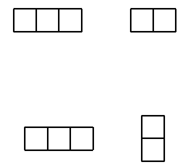


Figure 1.5: Showing G and H are equal

win, then G and H are equal.

Consider the two games of Domineering shown in Figure 1.5. The games G and H are clearly not the same, but if we consider G + (-H) we see that each player can make exactly one legal move. Therefore, whoever goes first will not be able to play on their next turn. This game is a second player win so it has value 0. We conclude that $G = H$ in this case. Note that because equality is a defined relation, it will be possible to have games which are not the same but which have the same value.

The above result can also be generalized so that it applies to other relations between G and H. The proofs of these statements are similar in nature and hence omitted. Therefore, we find that:

- $(G > H) \Leftrightarrow (G - H > 0)$
- $(G < H) \Leftrightarrow (G - H < 0)$
- $(G || H) \Leftrightarrow (G - H || 0)$

1.6 Canonical Form of a Game

As previously mentioned, games can take on the same value even when they are not identical. Fortunately, every game has a simplest canonical form which is identical for every game which shares the same value. We may simplify games through two different methods which result in the unique representation.

Definition 1.6.1 *Given a game* $G = \{G^{L_1}, G^{L_2}, ... | G^{R_1}, G^{R_2}, ...\}$, *a Left option* G^{L_1} *is* ***dominated*** *if there exists another Left option* G^{L_2} *such that* $G^{L_2} \geq G^{L_1}$. *Likewise,* *a Right option G^{R_1} is **dominated** if there exists another Right option G^{R_2} such that $G^{R_2} \leq G^{R_1}$.*

Lemma 1.6.1 (Deleting Dominated Options) *Given a game $G = \{A, B, ...|X, Y, ...\}$ along with $B \geq A$ and $Y \leq X$, then $G = \{B, ...|Y, ...\}$. In other words, the value of a game does not change when some or all of its dominated options are removed.*

Proof: First, let $G = \{A, B, ...|X, Y, ...\}$ and $H = \{B, ...|Y, ...\}$. Now, using our definitions we'd like to show that $G = H$. That means we need to show that $G \geq H$ and $H \geq G$.

We know that $G = G$ and $H = H$ which implies that $Y \triangleright H$. We also know that $Y \leq X$. Thus $X \triangleright H$ and there is no Right option of G which is $\leq H$. Also, the Left options of H are a subset of those of G so there is no Left option of H which is $\geq G$. Therefore, $G \geq H$.

Similarly, we know that $B \triangleleft H$ and $B \geq A$. This implies $A \triangleleft H$ and there is no Left option of G which is $\geq H$. Also, the Right options of H are a subset of those of G so there is no Right option of H which is $\leq G$. Therefore, $H \geq G$ and we can conclude that $G = H$. ■

Definition 1.6.2 *Let $G = \{G^L|G^R\}$ be a game. A Left option G^L is **reversible** if G^L has a Right option G^{LR} where $G^{LR} \leq G$. Likewise, a Right option G^R is reversible if G^R has a Left option G^{RL} where $G^{RL} \geq G$.*

Lemma 1.6.2 (Bypassing Reversible Options) *Given a game $G = \{H, G^L|G^R\}$, if H is a reversible option then $G = \{H^{RL}, G^L|G^R\}$. In other words, we may replace a reversible move H with all of the Left options of H^R.*

Proof: Let $G = \{H, G^L|G^R\}$ and $K = \{H^{RL}, G^L|G^R\}$. We need to show that $G = K$. Consider the game $G + (-K)$. If Left moves to $G + (-G^R)$, Right can make the corresponding move to $G^R + (-G^R) = 0$ and win. If Left moves to $G^L + (-K)$, Right can move to $G^L + (-G^L) = 0$ and win again. Finally, if Left moves to $H + (-K)$ Right responds by moving to $H^R + (-K)$.

From this point if Left moves to $H^{RL} + (-K)$ Right moves to $H^{RL} + (-H^{RL}) = 0$ and wins. Otherwise, Left must move to $H^R + (-G^R)$. In this case we note that since $H^R \leq G$ we know that $H^R + (-G) \leq 0$. This is the same as saying that Right will win despite any move Left can make to $H^R + (-G^R)$. Therefore, Left loses when he plays first and we get that $G \leq K$.

Now, following our definitions, since $K = K$ we know that there is no $G^R \leq K$. Since H is reversible we know that $G \geq H^R$. Also, we know that no $H^{RL} \geq H^R$. Therefore, there is no $H^{RL} \geq G$. Furthermore, there is no $G^L \geq G$ since we know that $G = G$. Thus we satisfy the condition to say that $G \geq K$.

We can therefore conclude that $G = K$. ■

Now, to arrive at the simplest form of the game we can repeatedly remove dominated options and replace reversible options as described above. We still must show that when there are no further dominated or reversible options, we have arrived at a unique representation for the game.

Theorem 1.6.3 *Every game has a unique representation with no dominated or reversible options.*

Proof: Suppose $G = H$ and they are both written in a form with no dominated or reversible options. Since $G - H = 0$, Right must have a winning move from $G^L - H$. It cannot be in G^L since this would then imply that $G^{LR} - H \leq 0$ and thus $G^{LR} \leq G$ telling us that G^L is reversible. Hence Right's winning move must be in $-H$ and we get $G^L - H^L \leq 0 \Leftrightarrow G^L \leq H^L$ for some H^L. Similarly, $H^L \leq G^{L'}$ for some $G^{L'}$. But then we know that $G^L \leq G^{L'}$ and thus $G^L = G^{L'}$ otherwise G^L is dominated. This finally gives us:

$$G^L \leq H^L \leq G^L \Leftrightarrow G^L = H^L$$

Therefore, every Left option of G is also a Left option of H. By symmetry we can see that G and H must have exactly the same sets of Left and Right options. ■

1.7 Numbers

Let $\mathcal{D}$ be the ring of dyadic rationals. That is, all numbers of the form $\frac{a}{2^k}$ where a, k are integers.

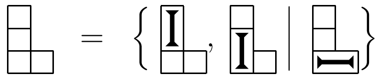
Definition 1.7.1 *For all $x \in \mathcal{D}$ we define a game $G(x)$ recursively according to the rules:*

- $G(0) = \{ \mid \}$
- $\forall n \geq 1, G(n) = \{G(n-1) \mid \}, G(-n) = \{ \mid G(-n+1)\}$
- $\forall k \geq 1, a$ *odd* $, G\left(\frac{a}{2^k}\right) = \left\{G\left(\frac{a-1}{2^k}\right) \,\middle|\, G\left(\frac{a+1}{2^k}\right)\right\}$

The games $G(x)$ are finite numbers which are usually written without the $G()$. To show that these games share the same properties as the numbers we are familiar with, we need to demonstrate games form a group and that $G(x+y) = G(x) + G(y)$ to conclude that G is a group homomorphism. These proof are omitted but can be found in [6].

The intuitive notion of a game as a numbers is how many moves advantage there is to the Left player. When we deal with integers, this is very easy to see. The game $0 = \{ \mid \}$ is of no advantage to either player. The game $1 = \{0 \mid \}$ is a one move advantage for Left. Similarly, negative game values will represent the number of moves advantage a game has for Right. This fits nicely with our notion of addition. If we have the compound game 5 + (-3), Left has a 5 move advantage in one component while Right has a 3 move advantage in the other. Since they alternate turns, Left will have a 2 move advantage left over when Right runs out of moves. Therefore, this compound game has a value of 2.

Of course, we also have games like the one in Figure 1.6 with value $\frac{1}{2} = \{0 \mid 1\}$. We would like to have an intuitive notion of what it means for Left to have half a



$$
\begin{aligned}
&= \{-1, 0 \mid 1\} \\
&= \{0 \mid 1\} \quad \text{(since -1 is dominated by 0 for Left)} \\
&= \frac{1}{2}
\end{aligned}
$$

Figure 1.6: A Domineering game with value $\frac{1}{2}$

move advantage. In particular, it is a game such that when there are two copies, it acts like one move advantage for Left. To verify this, we could check that the game $\frac{1}{2} + \frac{1}{2} - 1 = 0$ by showing that this game in three components is a second player win.

We can now use an alternate definition for birthday that makes use of the canonical form:

Definition 1.7.2 *The **birthday** of a game is one more than the highest birthday of its options in canonical form. We define the game* $0 = \{\ \mid\ \}$ *to have been born on day 0.*

Definition 1.7.3 *The **simplest** number given a particular condition is the number with the lowest birthday which satisfies that condition.*

Theorem 1.7.1 (Simplicity Rule) *If there is some number* x *such that* $G^L \triangleleft x \triangleleft G^R$*, then* $G = z$ *such that* z *is the simplest number satisfying* $G^L \triangleleft z \triangleleft G^R$*.*

Proof: To show $G = z$, we can equivalently show that $G - z = 0$. If Right plays first, she could move to $G^R - z$, but this is $\triangleright 0$ by our choice of z, so Left will win. Therefore, Right would have to move to $G - z^L$. Again we find that by our choice of z we know that $z^L \leq G^L$ or $z^L \geq G^R$. The latter is impossible since we know that $z^L < z \triangleleft G^R$. Therefore $z^L \leq G^L$ so Left can win by moving to $G^L - z^L \geq 0$. ■

The simplicity rule gives us a quick way to find the canonical form of a game. It says that if a number fits between the options of a game, then the game is equal to the simplest such number.

1.8 Switches

Now that we have a clear understanding of what numbers are, we can examine some games where there is no z such that $G^L \lhd z \lhd G^R$. Consider the Domineering position shown in Figure 1.7. In this case, the Left option is greater than the Right option. Given this position, both players would prefer to play first since it would give them a free move. Clearly, this game is in the N outcome class.

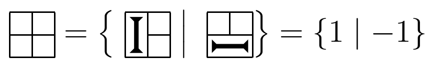


Figure 1.7: A Domineering position which is a switch

Definition 1.8.1 *A* ***switch*** *is a game of the form* $\{x \mid y\}$ *where* x *and* y *are numbers and* $x \geq y$.

Previously, we have seen the game $\star = \{0 \mid 0\}$ which is the simplest switch. We'd like to know how to compare these games to the numbers we've already seen.

Theorem 1.8.1 (Number Avoidance Theorem) *Given a sum of games, one should never play inside a number unless there is nothing else to do.*

The proof of this theorem is rather technical and does not give any insight with respect to the games I will later present. A full proof can be found in [8].

Lemma 1.8.2 *Given a game* $G = \{x \mid y\}$ *where* x *and* y *are numbers and* $x \geq y$, *then for any number* z *we have that:*

$$\text{if } z > x \text{ then } z > G$$

$$\textit{if } z < y \textit{ then } z < G$$

$$\textit{if } y \leq z \leq x \textit{ then } z || G$$

Proof: For the first case, we examine the game $z - G$ and show that Left can always win. If Right plays first, by Theorem 1.8.1 her best move must be to $z - x > 0$ so she loses. If Left plays first he can move to $z - y > z - x > 0$ and therefore wins.

For the second case, we use the same argument to show $z - G$ always has a winning strategy for Right. Finally, when $y \leq z \leq x$ we can see that whoever plays first will win because Left can move to $z - y \geq 0$ and Right could move to $z - x \leq 0$. Therefore, in this case $z||G$. ∎

Lemma 1.8.3 *If x, y and z are numbers with $x \geq y$, then we get that*

$$\{x \mid y\} + z = \{x + z \mid y + z\}$$

Proof: The proof follows directly from Theorem 1.8.1 and the definition of disjunctive sums. ∎

Lemma 1.8.4 *If x and y are numbers with $x \geq y$, then*

$$\{x \mid y\} = u + \{v \mid -v\} = u \pm v$$

where $u = \frac{x+y}{2}$ and $v = \frac{x-y}{2}$. This is called ***centralizing*** *a switch.*

Proof: Follows directly from Lemma 1.8.3. ∎

The game $\{1 \mid -1\}$ is therefore abbreviated ± 1 in most cases. The game $\{2 \mid 0\}$ can be written as 1 ± 1.

Switches which have been centralized are always in the N class of games, whereas numbers are always in L, R or P. It should seem intuitive that both players would prefer to play in a component which is in N so as to have more free moves when the game becomes a sum of numbers. This is basically what Theorem 1.8.1 previously told us.

1.9 Infinitesimals

Definition 1.9.1 *A game G is **infinitesimal** if $-z < G < z$ for all positive numbers z.*

Clearly, one rather trivial infinitesimal game is 0. Another infinitesimal game we've already seen is $\star$. This can be seen by adding any number z to it. If z is positive, then $z + \star$ is always a win for Left. Going first, he can move to z and win. If Right goes first, we know her best move must be to z by Theorem 1.8.1. This is positive, so she loses. If we let z be a negative number we would similarly find that Right always wins. Therefore, $\star$ is an infinitesimal game.

There are many other games which are infinitesimal as well. Consider the game $\{0 \mid \star\} = \uparrow$, pronounced 'up'. It is easy to see that this is a positive game since Left will win if he goes first or second. We can also check, however, that it is strictly less than all positive numbers and hence an infinitesimal. The corresponding game $\{\star \mid 0\} = \downarrow$ is called 'down' and is a win for Right.

An interesting property we now discover is that $0, \star, \uparrow$ and $\downarrow$ are representatives from each of the outcome classes. Despite having only values less than all positive numbers and greater than all negative numbers, we can still achieve any of the outcomes we previously defined.

1.10 Nimbers

In the first section, we defined a game called Nim which is played with piles of beans. It is an impartial game. It is worthwhile to note that all impartial games are either in the P or N classes. If Left has a winning strategy going first in a particular impartial game, then Right could adopt the same strategy if she were to move first. Therefore, Left could not possibly win if he were to move second and thus the game is not in L. For the same reason, the game cannot be in R either. Since either player could win by playing first, the game must be in N. If neither player has a winning strategy by moving first, then the game is P. As we already know, the only game in P has value 0. We now describe the values which impartial games can take when it is a member of the N class.

Definition 1.10.1 *The **nimber** $\star k$ is defined as the game equivalent to a game of nim in which there is one pile with exactly k beans in it.*

When you have a game of nim with one pile of k beans, each player's options are to move to a single pile with any number of beans from 0 to $k-1$. Since these options are all nimbers, we can define nimbers recursively as follows:

$$\begin{aligned} 0 &= \{\,|\,\} \\ \star &= \{0 \mid 0\} \\ \star 2 &= \{0, \star \mid 0, \star\} \\ \star k &= \{0, \star, \star 2, ..., \star(k-1) \mid 0, \star, \star 2, ..., \star(k-1)\} \end{aligned}$$

Definition 1.10.2 *Following the Sprague-Grundy theory [10] of impartial games, a game which has value $\star n$ is said to have **$\mathcal{G}$-value** n. Every impartial game is equal to some $\mathcal{G}$-value k.*

Definition 1.10.3 *Let A be a set of nimbers and let $B = \{k : \star k \in A\}$. Then mex(A) is $\star i$ where i is the* **m***inimal* **ex***cluded non-negative integer of B.*

Theorem 1.10.1 (Mex-Rule) *Let G be a game where the options for both players are a set S of nimbers. Then $G = mex(S)$.*

Proof: Let G be as described. Let $\star k = \text{mex}(S)$. To show that $G = \star k$ we need to demonstrate that the first player may move to any nimber $\star i$ for $i < k$ and that the second player can force the first player to make such a move if he so desires.

The first part is obvious by the definition of mex. Every value less than k must be in S otherwise k would not have been the minimal excluded value.

If the first player chooses to move to a game with value $\star i$ where $i > k$ then by our definition of $\star i$, the second player can move to $\star k$ forcing the first player to move to some $\star j$ where $j < k$ on a subsequent turn. It should be noted that since we have specified that all combinatorial games must terminate after a finite number of positions, a player will not be able to indefinitely make moves to larger nimbers. Clearly, the first player cannot move to a game $\star k$ since this again would violate our definition of mex. Therefore, it must be the case that $G = \text{mex}(S)$. ∎

Intuitively, you can think of a game which has options to larger heaps as being like nim but allowing the players to sometimes add beans to a pile instead of taking them away. In this case, the previous theorem states that this is of no advantage since the other player can simply reply by taking away the beans you just added.

Definition 1.10.4 *The* ***nim-sum*** *of two non-negative integers is the exclusive or (XOR), written ⊕, of their binary representations. This is equivalent to binary addition without carrying.*

$$\begin{array}{rrcl} & 3 & \rightarrow & (011)_2 \\ \oplus & 5 & \rightarrow & (101)_2 \\ \hline & 6 & \leftarrow & (110)_2 \end{array}$$

Figure 1.8: Example of the nim-sum operation

Theorem 1.10.2 [4] *A game of nim is a second player win if and only if the nim-sum of the sizes of the remaining heaps is 0.*

Proof: A game with no beans has nim-sum 0. We then note that when the nim-sum is 0, the next player must move to a position where the nim-sum is not 0 (as long as he has a legal move). When a player removes beans from a particular heap it must change the heap's binary representation which in turn must change the nim-sum of all the heaps.

Now we must show that the second player always has a good move when the nim-sum

$$
\begin{array}{rrcl}
 & 9 & \rightarrow & (1001)_2 \\
 & 4 & \rightarrow & (0\mathbf{1}00)_2 \\
 & 11 & \rightarrow & (1011)_2 \\
\oplus & 3 & \rightarrow & (0011)_2 \\
\hline
 & 5 & \leftarrow & (0101)_2
\end{array}
\quad \rightarrow \quad
\begin{array}{lccl}
(1001)_2 & \rightarrow & 9 & \\
(\mathbf{0001})_2 & \rightarrow & 1 & \\
(1011)_2 & \rightarrow & 11 & \\
(0011)_2 & \rightarrow & 3 & \oplus \\
\hline
(0000)_2 & \rightarrow & 0 &
\end{array}
$$

Figure 1.9: Finding a winning move in a game of nim with four piles

is not 0. To find a good move, we simply examine the binary representations of the heaps. Choose to play in a heap which has a 1 in the highest order magnitude where the nim-sum also generates a 1. Reduce this pile so that there are an even number of 1's in every order of magnitude, thus creating a nim-sum of 0.

This is a winning strategy for the second player when the initial nim-sum is 0. Also, by adopting the same strategy, the first player will be able to win any game in which the initial position does not have a nim-sum of 0. ■

This theorem can be generalized to show that if $G = H + K$ is the disjunctive sum of two impartial games, then $\mathcal{G}(G) = \mathcal{G}(H) \oplus \mathcal{G}(K)$ (see [6] or [2]).

1.11 Taking and Breaking Games

Taking and Breaking games ([5] and [2]) are impartial games played with heaps of beans. The players alternately choose a heap, remove a positive number of beans from that heap and then possibly splitting the remainder into several heaps. The number of beans to be removed and the number of heaps that one heap can be split into are given by the rules of the game.

An **octal game** is a taking and breaking game whose rules are specified by the octal code $\mathbf{d}_0.\mathbf{d}_1\mathbf{d}_2...\mathbf{d}_u$ where $0 \leq \mathbf{d}_i \leq 7$. If $\mathbf{d}_i = \mathbf{0}$, the player cannot remove i beans from any heap. If $\mathbf{d}_i = \delta_2 2^2 + \delta_0 2^0 + \delta_0 2^0$ where $\delta_j \in \{0, 1\}$, a player is allowed to remove i beans from a given heap provided that he split the remainder into exactly

j non-empty heaps where $\delta_j = 1$. Note that in such games we can never split a heap into more than 2 heaps.

A **subtraction game** is a specific type of octal game in which the players can never split a heap after removing beans. In particular we can use the same definition as above except that $\mathbf{d}_i \in \{\mathbf{0,3}\}$ for all i. The set $S = \{i : \mathbf{d}_i = \mathbf{3}\}$ is called the **subtraction set** for a particular subtraction game.

It is easy to see that Nim is a subtraction game (and hence an octal game) since splitting the heap is never allowed and players may take any number of beans from a heap. Therefore, it is represented as the octal game **0.3333...**.

When playing an octal game in which there are multiple heaps, we can determine the $\mathcal{G}$-value of the game by simply taking the nim-sum of the $\mathcal{G}$-values of the individual heaps. Therefore, to know everything about the game we simply need to know everything about games which begin with one heap. For a given game, let $\mathcal{G}(i)$ be the $\mathcal{G}$-value of the game played with a heap of size i.

Definition 1.11.1 *The* ***$\mathcal{G}$-sequence*** *for a particular octal game is said to be the sequence* $\mathcal{G}(0), \mathcal{G}(1), \mathcal{G}(2), ...$

Definition 1.11.2 *A $\mathcal{G}$-sequence is said to be* ***periodic*** *if there exist N, p such that $\mathcal{G}(n+p) = \mathcal{G}(n)$ for all $n \geq N$.*

The game **0.337** is an example of an octal game where a player may make the following moves:

- remove 1 or 2 beans from a heap and not split the remainder
- remove exactly 3 beans from a heap and can elect to split the remaining beans into two heaps.

Theorem 1.11.1 *Let a general octal game whose code digits $\mathbf{d}_z = \mathbf{0}$ for $z > t$ be given. If the $\mathcal{G}$-sequence is observed to have a period of length p after the last irregular* *value $\mathcal{G}(i)$, then the last value that needs to be computed to verify that the period persists is $\mathcal{G}(2i+2p+t)$.*

Proof: To show that $\mathcal{G}(2i+2p+t)$ is the last value that needs to be checked, we must show that the set of options when there are $2i+2p+t+1$ beans is the same as the set of options when there are $2i+p+t+1$ beans.

Let's say that we decide to remove exactly j beans. We know that $(t-j) \geq 0$ since t is the most we can legally remove on a player's turn. We may also have the opportunity to split the pile, making our options the set $\{\mathcal{G}(k) \oplus \mathcal{G}(2i+2p+t-j+1-k) : 0 \leq k \leq 2i+2p+t-j+1\}$.

We can assume without loss of generality that $k \leq 2i+2p+t-j+1-k$ and therefore $k \leq i+p$. That is to say, k represents the smaller heap if we choose to split it. Since we know that $0 \leq k \leq i+p$ then we find that

$$\begin{aligned}
2i+2p+t-j+1-k &\geq 2i+2p+1-k \\
&\geq i+p+1 \\
&> i+1 \\
&> i
\end{aligned}$$

Which means at most one of the piles falls into the range that may contain irregular values. Also, since $i+p+1$ and $i+1$ are both larger than i and differ by exactly p, we know that $\mathcal{G}(i+2p+1) = \mathcal{G}(i+p+1)$.

Therefore, it must be the case that:

$$\begin{aligned}
\mathcal{G}(2i+2p+t+1) &= \text{mex}\{\mathcal{G}(k) \oplus \mathcal{G}(2i+2p+t-j+1-k)\} \\
&= \text{mex}\{\mathcal{G}(k) \oplus \mathcal{G}(2i+p+t-j+1-k)\} \\
&= \mathcal{G}(2i+p+t+1)
\end{aligned}$$

Which tells us that the sequence of $\mathcal{G}$-values must be periodic for all larger heaps. ■

We can now use this theorem to show that the game **0.337** is periodic. We begin by simply calculating the $\mathcal{G}$-values which form its $\mathcal{G}$-sequence:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$\mathcal{G}(n)$	0	1	2	3	0	1	2	3	0	1	2	3	0	1

Table 1.2: $\mathcal{G}$-sequence for the octal game **0.337**

Notice that it seems to have a period of length 4 and no irregularities. Therefore, using the previous theorem we see that $t = 3, i = 0$ and $p = 4$. Thus, we must be sure to check up to and including $\mathcal{G}(2(0) + 2(4) + 3) = \mathcal{G}(11)$. Since our table checks at least this far, we can be sure that the pattern will continue indefinitely.

Chapter 2

Current Work

The games that I will examine in this thesis tend to take on only a subset of all possible game values. The following definitions and results are original work and are motivated by my desire to classify the game values which arise in the games I am studying. *Vertex Deletion* and *Grand Left/Right* have a common property that a move takes up exactly one 'space' which remains in the game. It is this property that prohibits many game values from occurring.

2.1 Introduction to the games

I will now give a brief overview of each of the games I will examine including their origins and rules for playing.

2.1.1 Vertex Deletion

In this collection of games, Left and Right alternately remove vertices (subject to some constraints) from a graph. All incident edges are also removed to produce an induced subgraph of the original. When a player is unable to make a legal move, they lose the game. Figure 2.1 shows a typical move in the game where a player removes the vertex labelled c.

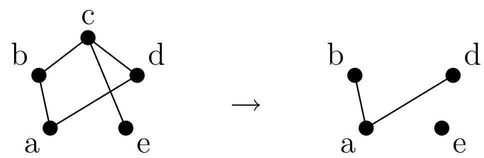


Figure 2.1: Typical move in the vertex deletion game

I will examine the game when played on both undirected and directed graphs. For each type of graph there are three variations for the game based on which vertices (even or odd degree) players may remove. I will show that some of these variations actually produce a strict subset of all game values while other variations have ties to octal games discussed earlier. There will also be discussion of specific classes of graphs and the game values they produce. In particular we will find graphs with values that form arithmetic periodic sequences with respect to the number of vertices they contain. This game originated at the Games At Dal conference which took place at Dalhousie University in August of 2002.

2.1.2 Grand Left/Right

Grand Left/Right is a game played by two players on a square grid of a predetermined size. It is a partizan game and each player has their own tokens which are black for Left and white for Right. The players decide on a starting position which involves placing 2 to 4 tokens on the board in an aesthetically pleasing way before play begins. On each player's turn, they select one of the pieces they currently have

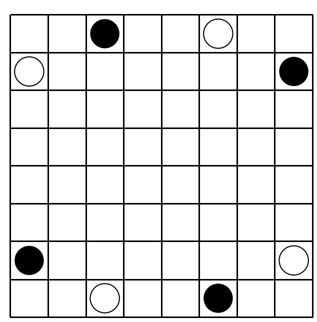


Figure 2.2: Possible starting position for Grand Left/Right

on the board and a cardinal direction for it to 'shoot'. The player then marks squares in that direction until the next square is occupied by another token, the next square is one you've already marked or you've reached the end of the board. At that point you turn 90 degrees and continue. At some point you will reach a square where after turning once you still cannot move. Place a new token on this square and erase all the marked squares. This ends that player's turn. In this version of the game, Left always turns his tokens to the right, and Right always turns her tokens to the left. The game ends when a player cannot make a legal move and thus loses.

Here is an example where Left plays first and chooses his leftmost token to 'shoot' upward. Its first turn is due to a white token blocking its path. The second turn is because of the edge of the board. Many of its final turns are because it cannot travel over a square it has previously occupied.

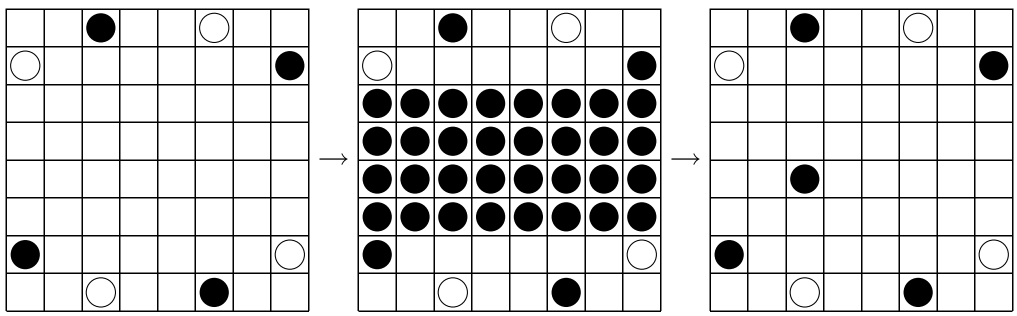


Figure 2.3: Left makes the first move in a game of Grand Left/Right

The original version of this game was played in the same manner except that a new piece was placed on every space along the path. In Figure 2.3 the first move would have ended with the board looking like the middle diagram. The version discussed in this thesis is more 'playable' and has the added feature of positions which can be classified as odd and even much like the Vertex Deletion game. This game was first posed as a problem by Prof. R. J. Nowakowski during a course in game theory in the fall of 2002 at Dalhousie University.

2.1.3 Cookie Cutter

In this game, we begin with an $x \times y$ grid and a fixed $k \times k$ cookie cutter (which we will refer to as being of size k). On each player's turn they place the cookie cutter so that it covers at least one of the squares of the grid (that is, it may overlap the edges of the grid). Then all squares (blocks) of the grid covered by the cookie cutter are removed and play continues with the next player. Naturally, the grid may not always remain a rectangle during this process. Under normal play, whoever takes the last block of the grid will win since the next player will not be able to make a legal move. Since the options of both players are identical we know this is an impartial game and therefore all the values of positions will be nimbers as described in the first chapter.

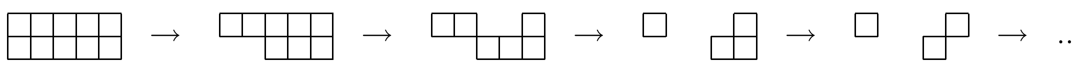


Figure 2.4: A possible sequence of moves with a cookie cutter of size 2

We will later show that this game has ties to octal games for which there is a great deal already known. This game originated in a problem solving course offered at the University of Waterloo during the summer of 2002.

2.2 New Definitions and Results

Definition 2.2.1 *Let E_0 and O_0 be the set of all even and odd integers respectively. Let E_k be the set of all games whose options are in O_{k-1}. O_k is the set of all games whose options are in E_{k-1} with the restriction on the options that the game 0 is not defined in O_k for any k. An **even game** is any game in the set E_k for some k while an **odd game** is any game in the set O_k for some k.*

The restriction given for the sets O_k are a natural consequence of the games under consideration. For example, in the Vertex Deletion game where players may delete vertices of opposite parity we will find that games with value 0 can never occur when there are an odd number of vertices left in the graph.

Lemma 2.2.1 *$E_k \subset E_{k+1}$ and $O_k \subset O_{k+1}$ for all k.*

Proof: We will proceed by induction. First we see that every even integer can be represented as a game with only odd options: $2n = \{2n-1|2n+1\}$. Likewise, every odd number can be represented as a game with only even options: $2n+1 = \{2n|2n+2\}$. Therefore, we know that $E_0 \subset E_1$ and $O_0 \subset O_1$.

Now, assume the statement is true up to k. Since $O_{k-1} \subset O_k$ then we know that the games in $E_k \subset E_{k+1}$ since we generate the games in E_{k+1} by taking all possible games formed by taking options from O_k. A similar argument also holds when we consider O_{k+1}. ■

Theorem 2.2.2 *The sets E_k and O_k are disjoint for all k.*

Proof: Assume there is as a game G that is both even and odd. Let k be the smallest value such that G is in both E_k and O_k. Then we can write $G = \{H_1 | H_2\} = \{K_1 | K_2\}$ where $H_1, H_2 \in O_{k-1}$ and $K_1, K_2 \in E_{k-1}$.

Now, we find the simplest form of G by removing its dominated and reversed options using both representations. If a left or right option of G is dominated, it is simply removed. If a left option is reversed, it is replaced by the left options of right's best response. Therefore, if the left option is an even game, it is again replaced by options which are even games. The same holds when we deal with odd games. Likewise if a reversed right option is an even game, it is replaced by the right options of left's best response. As before, if the right option is an even game, it is replaced by options which are even games (and again, this holds for odd games as well).

After performing these operations to both of G's representations we know that both new representations must have the exact same options since we proved earlier that this simplest form of a game is unique. Let these two simplest forms be $\{P|Q\}$. But by our process we know that any element of P or Q must be in E_{k-1} and O_{k-1} which would contradict that k is the smallest value such that there there is a game in both E_k and O_k. Therefore, we have that P and Q are both the empty set. But then we find that $G = \{\,|\,\} = 0$ which contradicts the fact that 0 never occurs in O_k for any k. Therefore, such a game G cannot exist and the sets E_k and O_k are disjoint for all k. ■

It should be noted that examining the simplest form of a game involves deleting and bypassing options available to both players. As demonstrated in the previous proof, this does not affect the property of a game being even or odd. The following results make use of this fact.

Lemma 2.2.3 *If a game G is infinitesimal and positive then*

- *Left can eventually move to a 0 regardless of whether he plays first or second.*
- *Left is unable to move to any positive number if Right plays optimally.*

Proof: Left will win this game so he, at some point, must move to a position with value ≥ 0. Assume for a contradiction that Left can at some point move to a game equal to a number $z > 0$. Then consider the game $G - \frac{z}{2}$. Left will still win this game since he can move to $z - \frac{z}{2} > 0$ which implies $G \triangleright \frac{z}{2}$ and therefore contradicts that G is infinitesimal. ∎

Lemma 2.2.4 *Let G be an infinitesimal game which is fuzzy with 0 but not equal to $\star$. Then G is not an even or odd game.*

Proof: Assume for a contradiction that G is an even or an odd game. We may assume that G is the game with the earliest birthday such that this is true. We will examine Left's options in this game. Similar arguments can be made for the Right options.

Case 1: Left has an option which is a positive number. That is, $G^L = z > 0$. As in Lemma 2.2.3, we can see that $G - \frac{z}{2} \triangleright 0$ which implies that G cannot be infinitesimal.

Case 2: G has a positive infinitesimal Left option. That is, $0 < G^L < z$. Then by Lemma 2.2.3 we know a 0 position can be reached by playing first or playing second. Recursively, this tells us either G has options where one is even and another is odd or G has an option from which a position of value 0 can be reached by playing first or second. At some point we reach a game which is not even or odd which tells us that G cannot be even or odd.

From cases 1 and 2 we conclude that Left has no options which are positive. We know that since $G||0$, Left can win going first. Since he has no options greater than 0, he must have an option equal to 0.

Case 3: Left has negative options. That is, $G^L < 0$. This cannot occur because these options are dominated (Left must have an option to 0) and hence removed.

Case 4: Left has an option which is fuzzy with 0. That is, $G^L||0$. If this option is $\star$ then G is not an even or odd game since he has options to 0 and $\star$ which are even and odd respectively. If the option is not $\star$, then we must have a game with an earlier birthday which satisfies the above properties. This cannot occur due to our original assumption about G.

Finally, if G has no other options besides 0, then we find that $G = \{0 \mid 0\} = \star$ which is again a contradiction. Therefore, such a game G does not exist. ■

Theorem 2.2.5 *The sets E_k and O_k do not have any games with values that contain fractions or any infinitesimals besides 0 and $\star$.*

Proof: By Lemmas 2.2.3 and 2.2.4 we know that infinitesimals besides 0 and $\star$ require that a 0 option exists at both an even and odd number of steps from the outset of the game. Therefore, we would have 0 in both E_k and O_k for some k which cannot happen. Thus, we cannot have infinitesimal values besides 0 and $\star$ which are even or odd.

Fractions in their simplest form only have other fractions or integers as their options (by our definition of numbers). Also, by our definition of simplicity (Definition 1.7.3), we know that if a game has a fractional value, its options can differ by at most 1. Recursively we can show that if a fraction has integer options it is not even or odd since the options must be consecutive and hence one option is even and the other is odd. If a fraction has options which are not both integers when in simplest form, then it must have a fractional option. This option is not even or odd, so the game itself is not even or odd. ■

As we've seen before, the game $\{0 \mid 1\}$ has value $\frac{1}{2}$, so you may be inclined to think that $G = \{0 \mid 1\star\}$ has the same value. In this case we see that if we play the game $\{0 \mid 1\star\} - 1$ we see that second player always wins and hence $G = \{0 \mid 1\star\} = 1$. The reason for this is that Right's option to $1\star$ is reversible and replaced by the empty set. Therefore, we get that $\{0 \mid 1\star\} = \{0 \mid \}$ which is the simplest form of the game 1.

We can construct rather strange looking games which satisfy the conditions of being even or odd. For instance, 3 and 1 are both odd integers so $\{3 \mid 1\}$ is an even game. Since 0 is also an even game, we can construct $\{\{3 \mid 1\} \mid 0\}$ which is an odd game (having only even games as options). It is values like these that appear in games like Vertex Deletion and Grand Left/Right.

Chapter 3

Vertex Deletion

It should be noted that throughout this section I will abuse notation and use G to refer to both a graph and the value of the game played on that graph.

First I will examine the game when played on an undirected graph. When both players remove only even degree vertices we will find that the game is completely solved. The version when both players remove only odd degree vertices has many positions with low $\mathcal{G}$-values but is not solved in general. The most interesting version is when Left removes even degree vertices and Right removes odd degree vertices. We will show that the graphs fall into the partition of even and odd games as discussed in chapter 2. We will also find that there are no graphs which have negative value and that some classes of graphs form sequences of game values. Finally, I will present a decomposition that allows for the simplification of some graphs when determining their value.

I will then proceed to analyze the game when played on directed graphs. In this case neither of the impartial versions are completely solved but we do find graphs which correspond to a particular octal game. In contrast to the game played on undirected graphs, I will show that we may have directed graphs with negative values when Left removes even in-degree vertices and Right removes odd in-degree vertices.

3.1 Undirected Graphs

3.1.1 Even/Even

In this version of the game, both Left and Right play on an undirected graph and can only remove vertices which have even degree. Since both players have the same options from a given position, this is an impartial game. Therefore, the values we can obtain are limited to 0 and the nimbers $\star n$ for any positive integer n.

Lemma 3.1.1 *If $|V(G)|$ is odd then $G \in N$.*

Proof: If $|V(G)|$ is odd then by Theorem $A.2.3$ there must be a vertex of even degree. Therefore, there exists some legal move in G. On each of the first player's subsequent turns there will again be an odd number of vertices. Thus, on each of their turns, a legal move exists. Since the first player will always have a legal move, the game can only end if the second player is unable to move. Since there are finitely many vertices in G eventually the second player will be unable to move causing them to lose. Therefore, the first player will always win regardless of the moves he makes throughout the game and thus $G \in N$. ■

Lemma 3.1.2 *If $|V(G)|$ is even then $G \in P$.*

Proof: If the first player has a legal move it must be to a graph with an odd number of vertices. So, by Lemma 3.1.1 the next player (which is the second player in the original game) will win. Since the first player can never win we get that $G \in P$. ■

Theorem 3.1.3 *When both players can remove only even degree vertices, the game is trivial and we have that:*

$$G = \begin{cases} 0 & \text{if } |V(G)| \text{ is even} \\ \star & \text{if } |V(G)| \text{ is odd} \end{cases}$$

Proof: By Lemma 3.1.2 we know that when $|V(G)|$ is even, the game has value 0. When $|V(G)|$ is odd, the only options available to each player are to games of value 0. Therefore, the game must have the form $G = \{0|0\} = \star$. ■

3.1.2 Odd/Odd

In this version of the game, both players may only remove vertices which have odd degree. Much like the Even/Even version, both players have the same options from a particular position which makes this an impartial game as well.

Lemma 3.1.4 *The path on n vertices, P_n has value* $\begin{cases} \star & \textit{if } n \textit{ is even} \\ 0 & \textit{if } n \textit{ is odd} \end{cases}$

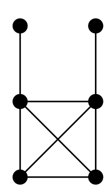
Proof: As long as $n > 1$ we know our path has exactly two vertices which are of degree 1 (the endpoints). When an endpoint is removed, we are left with a path with one fewer vertex. Therefore, on each player's turn a legal move must exist until there is a single vertex left over. The last vertex has degree 0 and hence cannot be removed by either player. Thus $P_1 = 0$. Since this is the only ending position for the game, when there are an even number of vertices the first player will win and when there are an odd number of vertices the second player will win. This gives us the values listed in the statement of the Lemma. ■

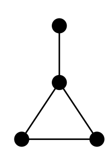
Here are a list of the values for games played on some other graphs:

K_n and S_n (star on n vertices) all have value $\begin{cases} \star & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$

The value of $K_{m,n}$ is $\begin{cases} \star & \text{if } n \text{ and } m \text{ are both odd} \\ 0 & \text{otherwise} \end{cases}$

Many graphs found thus far have value $\star$ or 0. At the time of writing, there are only 2 known graphs with value $\star 2$. They are shown here:





There are no known connected graphs that have value $\star n$ for any $n \geq 3$. We can however use the disjunctive sum of two connected graphs with values $\star$ and $\star 2$ to create a graph with value $\star 3$ (by the mex-rule).

3.1.3 Even/Odd

In this version of the game, Left may only remove vertices of even degree and Right may only remove vertices of odd degree. Since the players no longer have the same set of moves, this is a partizan game.

Lemma 3.1.5 *If $|G(V)|$ is odd, then $G \in L \cup N$.*

Proof: For $G \in L \cup N$ we must show that Left has a winning strategy by moving first. Since Left removes the even degree vertices, we know by Theorem A.2.3 that he always has a legal move when $|G(V)|$ is odd. On each of his subsequent turns, there will again be an odd number of vertices. Thus, after each of his opponent's turn he must have a legal move. Therefore, he must eventually make the last move and win the game. ∎

Lemma 3.1.6 *If $|G(V)|$ is even, then $G \in L \cup P$.*

Proof: If Right plays first, he must move to a position which has an odd number of vertices. By Lemma 3.1.5 this is in $L \cup N$ so Right will lose. Therefore, G cannot be in R or N. ∎

Corollary 3.1.7 *For any graph G, $G \not< 0$.*

Proof: By applying Lemmas 3.1.5 and 3.1.6 we can determine that all graphs are in $L \cup N \cup P$. In other words, $G \notin R$ and thus $G \not< 0$. ∎

Theorem 3.1.8 *If a graph G has an odd number of vertices, $G \in O_k$ for some k (G is an odd game). Likewise, if G has an even number of vertices, $G \in E_k$ for some k (G is an even game).*

Proof: The graph on 0 vertices has value 0 and is hence an even game. Also, the graph with 1 vertex has value 1 and is therefore an odd game. Assume that all graphs on $2k$ vertices are even games. Then we know that every option from a graph G on $2k+1$ vertices is even since a legal move for either player is to delete exactly one vertex. Also, by Lemma 3.1.5, we know $G \neq 0$. Therefore G is an odd game. Now, we know that all graphs on $2k+1$ vertices are odd games so a graph on $2k+2$ vertices can only have odd options. Therefore, it must be an even game. By induction, we now know that all graphs with an odd number of vertices are odd games and all graphs with an even number of vertices are even games. ■

Theorem 3.1.9 *The complete bipartite graphs have the following values:*

$$K_{2n+1,2m+1} = 0$$

$$K_{2n,2m} = \begin{cases} 0 & \textit{if } n > 0, m > 0 \\ 2n & \textit{if } m = 0 \\ 2m & \textit{if } n = 0 \end{cases}$$

$$K_{2n,2m+1} = \begin{cases} \{2n|0\} & \textit{if } n > 0, m = 0 \\ 2m+1 & \textit{if } n = 0 \\ \star & \textit{otherwise} \end{cases}$$

Proof: Beginning with the graph $K_{2n+1,2m+1}$ we see that Left has no legal moves since all vertices will have odd degree which implies $K_{2n+1,2m+1} \notin L$. Also, this graph has an even number of vertices so by Lemma 3.1.6 it must be in $L \cup P$. Therefore, the game must be in P and has value 0.

Now, $K_{2n,2m} = \{K_{2n-1,2m}, K_{2n,2m-1}|\}$ when $n > 0$ and $m > 0$. But then we see that from either of Left's options, Right can move to $K_{2n-1,2m-1}$ which has value 0. Therefore, the first player can never win and this graph has value 0. If $n = 0$ or $m = 0$, then $K_{2n,2m}$ is just a set of isolated vertices which is the disjunctive sum of $2m$, respectively $2n$, games of value 1.

Finally, for the game $K_{2n,2m+1}$, if $n = 0$ this is a game with $2m+1$ isolated vertices which has value $2m+1$. If $n > 0$ we see that $K_{2n,2m+1} = \{K_{2n,2m}|\,K_{2n-1,2m+1}\}$. Right's option always has value 0 and Left's option has value 0 when $m > 0$ and has value $2n$ when $m = 0$. Therefore, $K_{2n,2m+1} = \{0|\,0\} = \star$ for $m > 0$ and $K_{2n,2m+1} = \{2n|\,0\}$ for $m = 0$. ∎

Definition 3.1.1 *An **arithmetic periodic** sequence with* period p *and* saltus s *has terms* t_n *such that* $t_{n+p} = t_n + s$.

Theorem 3.1.10 *The value of the games played on the paths* P_n *form an arithmetic periodic sequence with the following values:*

$$P_k = \begin{cases} \frac{k}{3} \pm 1 & \textit{if } k \equiv 0 \textit{ mod } 3 \\ \frac{k+2}{3} & \textit{if } k \equiv 1 \textit{ mod } 3 \\ \frac{k-2}{3} & \textit{if } k \equiv 2 \textit{ mod } 3 \end{cases}$$

Proof: We proceed by induction and begin with the base case:

$P_1 = \{0|\,\} = 1 = \frac{1+2}{3}$
$P_2 = \{|1\} = 0 = \frac{2-2}{3}$
$P_3 = \{2|\,0\} = 1 \pm 1 = \frac{3}{3} \pm 1$

Now we assume the sequence holds up to $3k$. For any path P_n, Left's options are of the form $P_i + P_j$ where $i + j = n - 1$, $i \geq 1$, $j \geq 1$. Right's only option is to P_{n-1}.

For P_{3k+1} we begin by calculating Left's options. We know that $i + j = 3k$. If $i \equiv 1 \pmod 3$ and $j \equiv 2 \pmod 3$ then by the induction hypothesis, this option has value

$$\frac{i+2}{3} + \frac{j-2}{3} = \frac{3k}{3} = k$$

If $i \equiv 0 \pmod 3$ and $j \equiv 0 \pmod 3$ then by the induction hypothesis, this option has value

$$\frac{i}{3} \pm 1 + \frac{j}{3} \pm 1 = \frac{3k}{3} = k$$

Right's only option is to P_{3k} which has value $k \pm 1$. Therefore $P_{3k+1} = \{k | k \pm 1\} = k+1 = \frac{(3k+1)+2}{3} = P_{3k-2}+1$ as required. (Note that $\{k \mid k \pm 1\} - (k+1) = 0$: If Left moves to $k-(k+1) = -1$, Right wins and if Right moves to $(k \pm 1)-(k+1)$ then Left responds to (k+1)-(k+1)=0 and wins.

For P_{3k+2} we begin by calculating Left's options. We know that $i + j = 3k + 1$. If $i \equiv 0 \pmod 3$ and $j \equiv 1 \pmod 3$ then by the induction hypothesis, this option has value

$$\frac{i}{3} \pm 1 + \frac{j+2}{3} = \frac{3k+3}{3} \pm 1 = (k+1) \pm 1$$

If $i \equiv 2 \pmod 3$ and $j \equiv 2 \pmod 3$ then by the induction hypothesis, this option has value

$$\frac{i-2}{3} + \frac{j-2}{3} = \frac{3k-3}{3} = k-1$$

Right's only option is to P_{3k+1} which has value $k+1$. Therefore $P_{3k+2} = \{(k+1) \pm 1, k-1 | k+1\} = k = \frac{(3k+2)-2}{3} = P_{3k-1}+1$ as required. (Note that $k-1 \leq (k+1) \pm 1$)

Finally, we examine P_{3k+3} and begin with Left's options. We know that $i+j = 3k+2$. If $i \equiv 1 \pmod 3$ and $j \equiv 1 \pmod 3$ then by the induction hypothesis, this option has value

$$\frac{i+2}{3} + \frac{j+2}{3} = \frac{3k+6}{3} = k+2$$

If $i \equiv 0 \pmod 3$ and $j \equiv 2 \pmod 3$ then by the induction hypothesis, this option has value

$$\frac{i}{3} \pm 1 + \frac{j-2}{3} = \frac{3k}{3} \pm 1 = k \pm 1$$

Right's only option is to P_{3k+2} which has value k. Therefore $P_{3k+3} = \{k+2, k \pm 1 | k\} = (k+1) \pm 1 = \frac{(3k+3)}{3} \pm 1 = P_{3k}+1$ as required (Note that $(k+2) > (k \pm 1)$). We have shown that if the sequence holds up to $3k$ then it holds up to $3k+3$ and therefore by induction it must hold for all n. ■

Corollary 3.1.11 *The value of games played on the cycles C_n also form an arithmetic periodic sequence with the following values:*

$$C_k = \begin{cases} \frac{k}{3} & \textit{if } k \equiv 0 \textit{ mod } 3 \\ \frac{k-4}{3} & \textit{if } k \equiv 1 \textit{ mod } 3 \\ \frac{k+1}{3} & \textit{if } k \equiv 2 \textit{ mod } 3 \end{cases}$$

Proof: Left's only option from C_n is to P_{n-1} and Right has no legal moves since every vertex has degree 2. So, $C_n = \{P_{n-1}|\}$. When P_{n-1} is an integer, we get that $C_n = P_{n-1} + 1$. This occurs when $n - 1 \equiv 1$ or 2 (mod 3). When $n - 1 \equiv 0$ (mod 3), $C_n = \{\frac{n-1}{3} \pm 1|\}$. We see that Left's only option is reversible to $\frac{n-1}{3} - 2$ which then gives us that $C_n = \frac{n-1}{3} - 1$. Therefore, C_n forms an arithmetic periodic sequence with a period of length 3 and a saltus of 1 and takes on the values given in the statement of the Corollary. ∎

Theorem 3.1.10 showed us that paths form an arithmetic periodic sequence. There are many games in [2] which are conjectured to become periodic when you add enough copies of 'simple' positions. With respect to this game, the next theorem shows that graphs which are similar to paths also become arithmetic periodic when enough vertices along a path have been added.

Definition 3.1.2 *The graph A_n is constructed as a path on $n-1$ vertices with another vertex adjacent to the second vertex along the path.*

Figure 3.1: The graphs A_5 and A_6

Theorem 3.1.12 *For $n \geq 6$, the values generated by the graphs A_n form an arithmetic periodic sequence:*

$$A_n = \begin{cases} (\frac{n-3}{3})\star & \textit{if } n \equiv 0 \quad (\textit{mod } 3) \\ \left\{\frac{n-1}{3} \,\middle|\, (\frac{n-4}{3})\star\right\} & \textit{if } n \equiv 1 \quad (\textit{mod } 3) \\ (\frac{n-2}{3}) & \textit{if } n \equiv 2 \quad (\textit{mod } 3) \end{cases}$$

Proof: We will proceed by induction and begin by calculating the values for A_n for $3 \leq n \leq 8$. The values obtained are:

n	3	4	5	6	7	8
A_n	$\{2 \mid 0\}$	0	$\{\{3 \mid 1\} \mid 0\}$	$1\star$	$\{2 \mid 1\star\}$	2

We see that the values for $n = 6, 7$ and 8 are as predicted. This will serve as our base case. Now, for the general case, we need to determine what each player's options are. Right can remove any of the three vertices with degree 1 or the vertex of degree 3. Left can remove any of the other vertices. This gives us the expression

$$A_n = \{A_k + P_{n-k-1} \mid A_{n-1}, P_{n-1}, 2 + P_{n-3}\} \text{ where } 3 \leq k \leq n-2$$

By Theorem 3.1.10, we know that $2 + P_{n-3} > P_{n-1}$ which in turn means $2 + P_{n-3}$ is dominated for Right and is therefore deleted. Assuming the pattern holds for all smaller graphs, we can also check that A_{n-1} dominates P_{n-1}:
If $n - 1 \equiv 0 \pmod 3$, $A_{n-1} = (\frac{n-4}{3})\star = \left\{\frac{n-4}{3} \;\middle|\; \frac{n-4}{3}\right\} < \left\{\frac{n+2}{3} \;\middle|\; \frac{n-4}{3}\right\} = \frac{n-1}{3} \pm 1 = P_{n-1}$.
If $n - 1 \equiv 1 \pmod 3$, $A_{n-1} = \left\{\frac{n-2}{3} \;\middle|\; \frac{n-5}{3}\star\right\} < \frac{n+1}{3} = P_{n-1}$.
If $n - 1 \equiv 2 \pmod 3$, $A_{n-1} = (\frac{n-3}{3}) = P_{n-1}$.

In each case, A_{n-1} is at least as good for Right as moving to P_{n-1}. Therefore, the option to P_{n-1} can be deleted.

Left's options can likewise be simplified but in a less obvious fashion. When removing a vertex from the center of the graph, we produce two components, P_i and A_j. Since $i + j$ is fixed and $P_i + A_j = P_{i+3} + A_{j-3}$ (both sequences being arithmetic periodic with periods of length 3 and saltus of 1) then we only only have three unique Left options that need to be considered. When $n \equiv 1$ or $2 \pmod 3$, we find that Left's option to $A_3 + P_{n-4}$ dominates all others:
If $n \equiv 1 \pmod 3$ his options are:

$$A_3 + P_{n-4} = \{2 \mid 0\} + \frac{n-4}{3} \pm 1 = \frac{n-1}{3}$$

$$A_{n-3} + P_2 = \left\{\frac{n-4}{3} \;\middle|\; (\frac{n-7}{3})\star\right\}$$

$$A_{n-2} + P_1 = \frac{n-4}{3} + 1 = \frac{n-1}{3}$$

We find that the largest is $A_3 + P_{n-4}$.

If $n \equiv 2 \pmod 3$ his options are:

$$A_3 + P_{n-4} = \{2 \mid 0\} + \frac{n-2}{3} = \frac{n+1}{3} \pm 1$$

$$A_{n-3} + P_2 = \frac{n-5}{3}$$

$$A_{n-2} + P_1 = \frac{n-5}{3} \star + 1 = \frac{n-2}{3}\star$$

Here we find that $A_3 + P_{n-4}$ is again the largest.

When $n \equiv 0 \pmod 3$ his option to $A_{n-2} + P_1$ dominates all others:

$$A_3 + P_{n-4} = \{2 \mid 0\} + \frac{n-6}{3} = \frac{n-3}{3} \pm 1$$

$$A_{n-3} + P_2 = \frac{n-6}{3}\star$$

$$A_{n-2} + P_1 = \left\{\frac{n-3}{3} \,\middle|\, \frac{n-6}{3}\star\right\} + 1 = \left\{\frac{n}{3} \,\middle|\, \frac{n-3}{3}\star\right\}$$

In this case, we see that $A_{n-2} + P_1$ is the largest.

Now we can write an expression for A_n with only one option for either player. Using induction we can now see that the graphs A_n produce exactly the sequence described above:

For $n \equiv 0 \pmod 3$ we get that

$$\begin{aligned} A_n &= \{A_{n-2} + P_1 \mid A_{n-1}\} \\ &= \left\{\left\{\frac{n-3}{3} \,\middle|\, \frac{n-6}{3}\star\right\} + 1 \,\middle|\, \frac{n-3}{3}\right\} \\ &= \left\{\frac{n-3}{3} \,\middle|\, \frac{n-3}{3}\right\} \qquad \text{(Left's option reverses)} \\ &= (\frac{n-3}{3})\star \end{aligned}$$

For $n \equiv 1 \pmod 3$ we get that

$$\begin{aligned} A_n &= \{A_3 + P_{n-4} \mid A_{n-1}\} \\ &= \left\{\{2 \mid 0\} + \frac{n-4}{3} \pm 1 \,\middle|\, \frac{n-4}{3}\star\right\} \\ &= \left\{1 \pm 1 + \frac{n-4}{3} \pm 1 \,\middle|\, \frac{n-4}{3}\star\right\} \\ &= \left\{\frac{n-1}{3} \,\middle|\, \frac{n-4}{3}\star\right\} \qquad \text{(since } (\pm 1) + (\pm 1) = 0\text{)} \end{aligned}$$

For $n \equiv 2 \pmod 3$ we get that

$$\begin{aligned} A_n &= \{A_3 + P_{n-4} \mid A_{n-1}\} \\ &= \left\{\{2 \mid 0\} + \frac{n-2}{3} \,\middle|\, \left\{\frac{n-2}{3} \,\middle|\, \frac{n-5}{3}\star\right\}\right\} \\ &= \left\{\left\{\frac{n+4}{3} \,\middle|\, \frac{n-2}{3}\right\} \,\middle|\, \left\{\frac{n-2}{3} \,\middle|\, \frac{n-5}{3}\star\right\}\right\} \\ &= \frac{n-2}{3} \qquad \text{(both players' options reverse)} \end{aligned}$$

To see the last equality, we can play the difference $A_n - (\frac{n-2}{3})$. After either player moves, the other can move to a position of value 0, so this is a second player win and the games are equal. ■

Definition 3.1.3 *The graph $P_{n,k}$ has n vertices, $n-k$ of which lie on a path, while the other k are all adjacent to only the second vertex in the path.*

The graph $P_{n,k}$ is only defined for $n > k$. When $k = 0$ we have the paths P_n that we are familiar with. When $k = 1$ we have the graphs A_n which are described above. Another way of thinking of these graphs is having the graph S_{k+1} (star on k+1 vertices) then extending a path from the vertex of degree k until there are n vertices in total. We've already seen that for $k = 0$ or 1 that these graphs form an arithmetic periodic sequence.

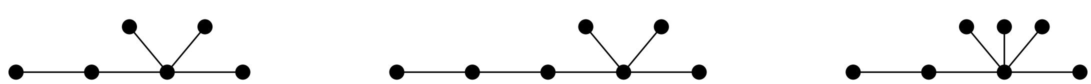


Figure 3.2: The graphs $P_{6,2}$, $P_{7,2}$ and $P_{7,3}$

Table 3.1 shows the game values for $n \leq 15$ and $k \leq 4$. We notice that for a fixed k and large enough n we see that the associated column of values seem to become arithmetic periodic. In particular, for k odd, the value of $P_{n,k}$ is the same as $P_{n-2,k-2}$. For k even and $k \geq 4$, we get that $P_{n,k}$ is the same as $P_{n-2,k-2}$ but with the Left option increased by 2.

$n \downarrow k \rightarrow$	0	1	2	3	4
1	1	-	-	-	-
2	0	0	-	-	-
3	$\{2 \mid 0\}$	$\{2 \mid 0\}$	$\{2 \mid 0\}$	-	-
4	2	0	0	0	-
5	1	$\{\{3 \mid 1\} \mid 0\}$	$\{4 \mid 0\}$	$\{4 \mid 0\}$	$\{4 \mid 0\}$
6	$\{3 \mid 1\}$	$1\star$	$\{3 \mid 1\}$	0	0
7	3	$\{2 \mid 1\star\}$	$\{\{5 \mid 3\} \mid 1\star\}$	$\{\{5 \mid 1\} \mid 0\}$	$\{6 \mid 0\}$
8	2	2	$\{5 \mid \{2 \mid 1\star\}\}$	$1\star$	$\{5 \mid 1\}$
9	$\{4 \mid 2\}$	$2\star$	$\{4, \{5 \mid 3\} \mid 2\}$	$\{2 \mid 1\star\}$	$\{\{7 \mid 5\} \mid 1\star\}$
10	4	$\{3 \mid 2\star\}$	$\{\{6 \mid 4\} \mid 2\star\}$	2	$\{7 \mid \{2 \mid 1\star\}\}$
11	3	3	$\{6 \mid \{3 \mid 2\star\}\}$	$2\star$	$\{6 \mid 2\}$
12	$\{5 \mid 3\}$	$3\star$	$\{5 \mid 3\}$	$\{3 \mid 2\star\}$	$\{\{8 \mid 6\} \mid 2\star\}$
13	5	$\{4 \mid 3\star\}$	$\{\{7 \mid 5\} \mid 3\star\}$	3	$\{8 \mid \{3 \mid 2\star\}\}$
14	4	4	$\{7 \mid \{4 \mid 3\star\}\}$	$3\star$	$\{7 \mid 3\}$
15	$\{6 \mid 4\}$	$4\star$	$\{6 \mid 4\}$	$\{4 \mid 3\star\}$	$\{\{9 \mid 7\} \mid 3\star\}$

Table 3.1: Values generated by the graphs $P_{n,k}$

Theorem 3.1.13 *For a fixed* k*, the graphs* $P_{n,k}$ *form an arithmetic periodic sequence. In particular, when* $k \geq 3$ *is odd,* $P_{n,k} = P_{n-2,k-2}$ *and when* $k \geq 4$ *is even,* $P_{n,k}$ *is the same as* $P_{n-2,k-2}$ *but with the Left option increased by 2.*

Proof: We will proceed by induction on both n and k and prove the result for the two cases, when k is even and when k is odd.

First we assume that k is odd and examine Right's options from the game $P_{n,k}$. She may remove any of the vertices of degree 1, or the vertex of degree $k+2$. This results in three different graphs which are $P_{n-1,k-1}$, $P_{n-1,k}$ and $(k+1)+P_{n-k-2,0}$. By induction, we know the value of all these graphs and find that $P_{n-1,k}$ is strictly the smallest. Therefore, her other two options are dominated. Also, we should note that by induction we know that $P_{n-1,k} = P_{n-1-2j,k-2j} = P_{n-k,1}$. Therefore, her best option has the same value as if she were playing in the game $P_{n-k+1,1}$. Left's options, on the other hand, are all of the form $P_{n-i-1,k} + P_{i,0}$, $1 \leq i \leq n-k-3$. Again, we know by induction that $P_{n-i-1,k} = P_{n-i-1-2j,k-2j} = P_{n-i-k,1}$. Therefore, his best move has exactly the value it would if he were playing in the graph $P_{n-k+1,1}$. Since we know that both players' best options have the same value as playing in $P_{n-k+1,1}$, then we know $P_{n-k+1,1} = P_{n,k}$ and $P_{n,k}$ forms an arithmetic periodic sequence.

For the second case, we assume that k is even. Since we will rely on the fact that $k-1$ is arithmetic periodic, the following results will hold only after the initial irregular values. Therefore, we require that $n \geq k+5$. From the graph $P_{n,k}$, Right's only options are to the graphs $P_{n-1,k-1}$ and $P_{n-1,k}$. By induction we know that $P_{n-1,k-1} = P_{n-1-2j,k-1-2j} = P_{n-k+1,1} < P_{n-k+1,2} < P_{n-1,k}$. Therefore, Right's best option has value $P_{n-k+1,1}$ which would be the same value as her option if she were playing in the game $P_{n-2,k-2}$ by our induction hypothesis. Now, Left's options are to $(k+1)+P_{n-k-2,0}$ and $P_{n-i-1,k}+P_{i,0}$. As we have seen when k was odd, $(k+1)+P_{n-k-2,0}$ is strictly larger than all other options even when $k=1$, so it will still be his best option here. We note that in the game $P_{n-2,k-2}$, his best option would have been to $(k-1)+P_{n-k-2,0}$ for exactly the same reason. Therefore, Left's option in the game $P_{n,k}$ is exactly 2 greater than in $P_{n-2,k-2}$ and Right's option is the same as in the game $P_{n-2,k-2}$ (ie $P_{n,k} = \{P_{n-2,k-2}+2 \mid P_{n-2,k-2}\}$). Therefore $P_{n,k}$ forms an arithmetic periodic sequence as described in the statement of the theorem. ∎

This gives further evidence to the fact that there are many graphs that form arithmetic periodic sequences when adding arbitrarily long paths extending from them.

Although we have completely solved the graphs $P_{n,k}$, the values themselves do not always tell us the optimal strategy. If we were to play a game which is a disjunctive sum of graphs of the form $P_{n,k}$ then we'd like to know where our best move lies. In this case, we note that for large n and k even, the difference between the Left and Right options increase as k increases. The greater the difference, the more inclined a player is to move in that component. Therefore, a player should choose to play in the component with the highest even value of k. When there are none left, we find that the components which have k odd and $n - k \equiv 0 \pmod 3$ are similar so switches and will therefore be preferred over all other moves. Finally, if there are no other moves left we find that the game is simply the sum of integers and stars. This may not provide the best strategy in all cases since we have ignored the graphs where $n - k \leq 4$. For these graphs, we find they may have greater differences in their Left and Right options than components with greater k values. For example, the in game $P_{5,3} + P_{6,2}$ we would prefer to play in the component $P_{5,3}$ despite k being odd.

The graphs discussed in the last three theorems are all special cases of trees. Table 3.2 shows the game values of all trees up to 7 vertices for this variant of the vertex deletion game.

3.1.4 Decompositions

A question of particular interest when examining a new game is to determine those positions which can be shown to be equivalent to smaller positions. These decompositions allow us to take a large graph and remove vertices and edges without affecting the value of the game. The decomposition shown here relates only to the Even/Odd variant of the vertex deletion game played on simple graphs.

Theorem 3.1.14 *Let G be a graph. Let $x, y \in V(G)$ such that $N[x] = N[y]$. Then the game played on G has the same value as the game played on $G' = G - \{x, y\}$.*

Proof: To show that $G = G'$ we can simply play the game $G - G'$ and show that the second player has a winning strategy. Since x and y have the same neighbourhoods, we can deduce that the parity of the degree of all other vertices is the same in G as it is in G'. We also know that there is an edge between x and y.

It is important to remember that the Left player will be able to remove even degree vertices in G but odd degree vertices in $-G'$ since the roles of the players will

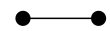
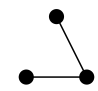
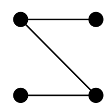
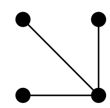
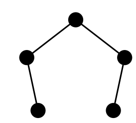
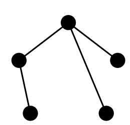
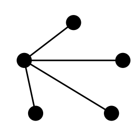
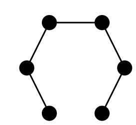
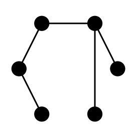
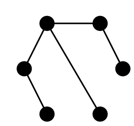
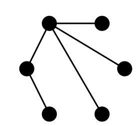
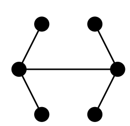
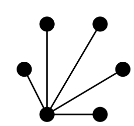
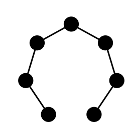
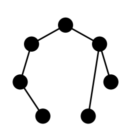
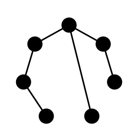
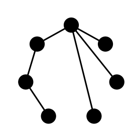
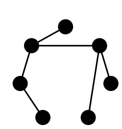
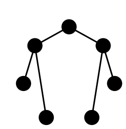
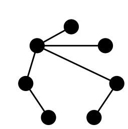
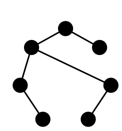
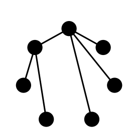
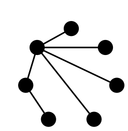
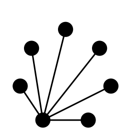
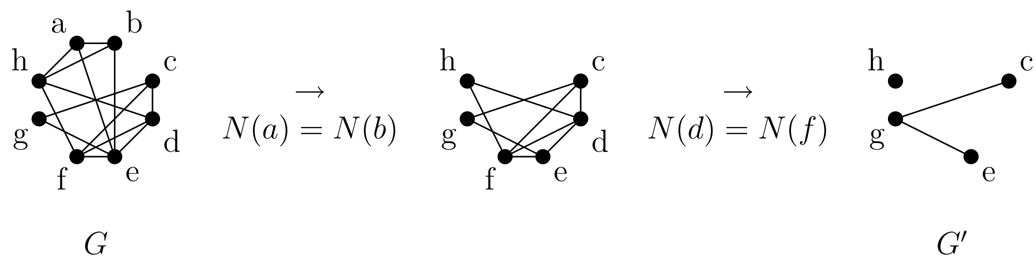
Graph	Value	Graph	Value	Graph	Value
	1		0		$\{2\|0\}$
	2		0		1
	$\{3\|1\|\|0\}$		$\{4\|0\}$		$\{3\|1\}$
	$1\star$		$\{3\|1\}$		$\{3\|1\}$
	0		0		3
	$\{2\|1\star\}$		$\{2\|1\star\}$		$\{5\|3\|\|1\star\}$
	$\{4\|2\|\|1\|\|\|0\}$		$\{2\|1\star\}$		3
	$\{2\|0\}$		$\{5\|3\|\|0\}$		$\{5\|1\|\|0\}$
	$\{6\|0\}$				

Table 3.2: All undirected tree game values up to 7 vertices for Even/Odd

be reversed. If the first player deletes a vertex which is not x or y in one game, the second player can certainly delete the equivalent vertex in the other game since the vertices have same parity of degree. On the other hand if the first player deletes x or y, then the second player can delete the other since its parity will have changed (due to deleting the edge between x and y). Therefore, regardless of the first player's moves, the second player always has a legal move to make in response. Since the game must eventually end we know it must be the second player who makes the last move. Thus, $G - G' = 0$ and we find that $G = G'$. ∎



$$G = G' = 1 + \{2 \mid 0\} = \{3 \mid 1\} = 2 \pm 1$$

Figure 3.3: Using the decomposition rule to evaluate a graph

This decomposition also gives us a way to play the original game rather than just calculate it's value. A player should adopt their best strategy as if playing on the simplified graph. If either player deletes a vertex that was removed during the simplification, then the other player should delete its pair that was removed in the same step of simplification. As long as both players are playing optimally, there will be no better move for either player to make.

We can use this theorem to solve other classes of graphs where vertices have the prescribed property. For instance, the complete graphs are such that every vertex has the same neighbourhood since all possible edges exist in the graph.

Theorem 3.1.15 *The complete graph K_n has values:*

$$K_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

Proof: Since every vertex has the same neighbourhood, we can remove any two at a time and achieve a graph with the same value. This of course generates the graph K_{n-2} after the first step. Therefore, all the complete graphs on an odd number of vertices have the same value as K_1 which is a single vertex with value 1. Also, the complete graphs with an even number of vertices are equivalent to the empty graph which has value 0. ∎

Theorem 3.1.16 *The graph $G_{n,k}$ is constructed by taking the graphs K_n and P_k then identifying the end vertex of P_k with an arbitrary vertex of K_n. Then $G_{n,k}$ has value:*

$$K_n = \begin{cases} P_k & \text{if } n \text{ is odd} \\ P_{k+1} & \text{if } n \text{ is even} \end{cases}$$

Proof: We know that every vertex that was originally in K_n besides the one which was identified has the same neighbourhood. Therefore, using our decomposition we can throw away pairs of vertices until we are left with either K_1 or K_2 (depending on whether n was odd or even). Now, K_1 (which is a single vertex) identified with P_k remains P_k. K_2 identified with P_k simply adds an extra vertex to the end of the path, so the resultant graph is P_{k+1}. ∎

This gives another example of a class of graphs which provide arithmetic periodic sequences of game values when we extend a path from one of its vertices.

3.1.5 Aside: An interesting class of graphs

During the course of investigating the values of games played in the Even/Odd variant on simple graphs, an interesting class of graphs was discovered. These are the complete bipartite graphs $K_{m,n}$ where $m = 1$. We call these the *stars* on $n + 1$ vertices, denoted S_{n+1}. Theorem 3.1.9 has completely solved these graphs but they are worth revisiting.

The stars on $2n$ vertices, S_{2n}, all have value 0. This is easy to see because there is one vertex of degree $2n-1$ and $2n-1$ vertices of degree 1. Since all vertices have odd degree, Left cannot make a move and loses if he plays first. Thus the game must be in $R \cup P$. Of course, we've shown that there are no graphs in R, so this must be in P and hence has value 0. This can be generalized by saying that all graphs where all vertices have odd degree have value 0. The argument is exactly as above.

Right can remove a vertex of degree 1 to produce the graph S_{2n-1} or remove the vertex of degree $2n-1$ to produce $2n-1$ isolated vertices. That game will have value $2n-1$. Therefore, the stars on $2n$ vertices have the following form:

$$S_{2n} = \{ \mid S_{2n} - 1, 2n - 1\}$$

To compute the exact value we need to know the value of S_{2n-1}. In that graph, Left has a very good move by taking the vertex of degree $2n-2$ leaving $2n-2$ isolated vertices. This game has value $2n-2$. Right's only option, on the other hand, is to S_{2n-2} which we've already shown has value 0. Therefore we get that $S_{2n-1} = \{2n-2 \mid 0\}$.

Substituting that back into our original equation, we get:

$$S_{2n} = \{ \mid \{2n-2 \mid 0\}, 2n-1\}$$

Now we find that Right's other option to $2n-1$ is dominated and is therefore removed. Right's other option is reversible and when simplified would show that this game has value 0 as we expected. The interesting thing is that if we actually play this game it is certainly a second player win. But, if Right must move first Left gets many 'free moves' since the value of the game will suddenly be very positive with value $2n-2$.

The term *zugzwang* is derived from German meaning 'compulsion to move' and is used by chess enthusiasts to describe a position in which passing would be preferable to any legal move. *Mutual zugzwang* is when neither player has a good move and would prefer to have the option to pass. With respect to combinatorial game theory, mutual zugzwang is a position which has value 0 despite the fact that players may still have legal moves to make. In some sense, the stars with an even number of vertices are even worse than mutual zugzwang since if it is Right's turn, she does not only lose the game, but she gives Left many extra free moves in the process.

3.2 Directed Graphs

We will now consider the same game as before, except that we will play on a directed graph (or digraph). The players will be able to remove vertices based on the parity of their in-degree. A dictionary of all digraphs on up to four vertices is given in the appendix along with the respective values for all three versions of the game.

3.2.1 Even/Even

We begin with the game where both players can remove vertices that only contain even in-degree. Naturally, this is an impartial game since both players have the same set of options from a given position.

Theorem 3.2.1 *Let D be arbitrarily directed forest with $|V(D)| = n$. Then we have that*

$$D = \begin{cases} \star & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

Proof: A forest is simply a union of trees. By Theorem A.2.4 we know that a tree with n vertices has exactly $n-1$ arcs. Therefore, a forest on n vertices will have exactly $n-c$ arcs where c is the number of components of D. So, the sum of the in-degrees over all vertices will be $n-c$. Since there are n vertices we know there must exist some vertex with in-degree 0. In particular, there is a vertex with even in-degree which implies that the first player has a legal move as long as $n \neq 0$. When a vertex and all incident arcs is deleted we cannot possibly create a new cycle in the digraph so we must be left with another directed forest with one fewer vertex. Therefore, there will always be a legal move for both players until there are no vertices remaining which makes this a trivial game with the values listed above. ∎

Unlike the game played on undirected graphs, this is not a trivial game in general. In other words, there are some positions that take on values which are not 0 or $\star$ (shown in Table 3.3). It is unknown whether or not there is any bound on the $\mathcal{G}$-values that these digraphs can take. At the time of writing, no connected digraphs that have any $\mathcal{G}$-value greater than 2 have been found. A large number of examples have been examined up to 7 vertices, but it was not an exhaustive search. A complete table of all digraphs on up to 4 vertices as well as the paths on 5 vertices along with their game values for all three variations can be found in Tables 3.3, 3.4 and 3.5 at the end of this chapter.

3.2.2 Odd/Odd

Theorem 3.2.2 *Let P_n be the digraph which is a path where all arcs are oriented in the same direction. The $\mathcal{G}$-value of the game played on P_n is the same as $\mathcal{G}(n)$ for the octal game* **0.6**.

Proof: Let's begin by examining the types of moves a player may make:

- remove a vertex and leave two graphs P_i and P_j where $i + j = n - 1 > 0$
- remove the end vertex to leave the graph P_{n-1} as long as $n - 1 > 0$

The rules are the same for each component when the graph becomes disconnected. Therefore, we can see that this is the same as playing a game with a heap of beans in which the legal moves are to remove 1 bean and leave one or two heaps but never take the last bean from any heap. Therefore, by the naming conventions for octal games given in section 1.11, this is the game **0.6**. ■

Among octal games, **0.6** is rather famous because it has such simple rules and yet does not seem to be periodic. There are many such 'badly behaved' octal games, and we shall examine others in chapter 5. Achim Flammenkamp has analyzed many 'badly behaved' octal games and maintains a website where this data can be found [7]. At the time of writing, **0.6** has been analyzed up to 2^{37} by J.P. Grossman [9] and there has not been any observed period. Figure 3.4 shows a graph of the first 30000 values of its $\mathcal{G}$-sequence.

Figure 3.4: The first 30000 values generated by the octal game **0.6**

3.2.3 Even/Odd

Like its counterpart played on simple graphs, this version of the game is partizan since the players have different options. I will examine a couple classes of graphs and determine their outcome class and values.

Theorem 3.2.3 *Let D be an arbitrarily directed forest with $|V(D)| > 0$. Then $D \in L$ and hence $D > 0$.*

Proof: As seen before, a directed forest with $|V(D)| > 0$ always has at least one vertex with in-degree 0. Therefore, Left always has a legal move. Thus, the only position which Left has no move would be when there are no vertices. But then we can deduce that Right must have made the previous move which would have been to delete a single vertex with no edges (and hence in-degree 0). Since this is not a legal

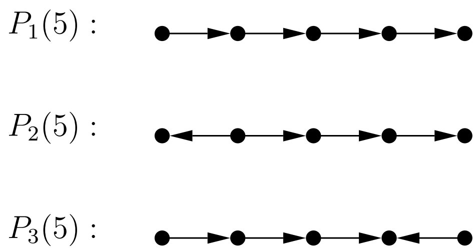


Figure 3.5: The digraphs $P_1(5)$, $P_2(5)$ and $P_3(5)$

move for Right, Left can never be in a position where he has no legal move. This tells us that Left can never lose, i.e. he will always win going either first or second. ∎

Definition 3.2.1 *The digraphs denoted $P_1(n)$, $P_2(n)$ and $P_3(n)$ are paths with n vertices where all arcs are oriented in the same direction with the exception that $P_2(n)$ has its first arc flipped and $P_3(n)$ has its last arc flipped.*

Lemma 3.2.4 *The digraph $P_1(n)$ has value:*

$$\begin{cases} 1\star & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$$

Proof: We will proceed by induction. When $n = 1$ the Left may remove it but Right cannot since it has in-degree 0. Therefore, $P_1(1) = \{0 \mid \} = 1$. When $n = 2$, either player may move to $P_1(1)$ since there is one vertex with in-degree 1 and one of in-degree 0. Thus we have $P_1(2) = \{1 \mid 1\} = 1\star$.

Now let's assume the pattern holds up to $P_1(k)$. For the game $P_1(k+1)$ Left can only remove the vertex with in-degree 0 leaving the game $P_1(k)$. Right can remove any other vertex leaving $P_1(k)$ (by taking the other end) or $P_1(i)+P_1(k-i)$ by taking the $(i+1)^{\text{th}}$ vertex along the path splitting it into two paths of the same type. Therefore,

$$P_1(k+1) = \{P_1(k) \mid P_1(k), P_1(i) + P_1(k-i)\} \quad \text{where } 1 \le i \le k-1$$

When we examine the right options we see that $P_1(i) + P_1(k-i) = 2$ or $2\star$. In either case, it is certainly greater than Right's other option of $P_1(k) = 1$ or $1\star$ so it is dominated.

So, for $k+1$ odd, we find that

$$P_1(k+1) = \{P_1(k) \mid P_1(k)\} = \{1\star \mid 1\star\} = 1$$

And for $k+1$ even we find that

$$P_1(k+1) = \{P_1(k) \mid P_1(k)\} = \{1 \mid 1\} = 1\star$$

■

Lemma 3.2.5 *The digraph $P_2(n)$, $n \geq 3$ has value:*

$$\begin{cases} \{2\star \mid 1\} & \text{if } n \text{ is even} \\ \{2 \mid 1\star\} & \text{if } n \text{ is odd} \end{cases}$$

Proof: As before, we will proceed by induction. For $n = 3$ we find that Left's only legal move is to a digraph with 2 vertices and no arcs. If Right plays first, both of her moves are equivalent and move to a digraph with 2 vertices and and arc between them. This game has value $1\star$. Thus, $P_2(3) = \{2 \mid 1\star\}$ as desired.

Now assume it the pattern holds up until $P_2(k)$. In the game $P_2(k+1)$, Left only has one legal move which creates a single vertex and a path $P_1(k-1)$. Right can remove any other vertex which may split the digraph into two components $P_2(i)$ and $P_1(k-i-1)$. Therefore, the Left and Right options from $P_2(k+1)$ may be described as follows:

$$P_2(k+1) = \{1 + P_1(k-1) \mid P_1(k), P_2(i) + P_1(k-i-1)\}$$

where $3 \leq i \leq k-1$. As before, we can now examine Right's options to eliminate ones which are dominated. First we note that $P_2(i)$ has value $\{2 \mid 1\star\}$ or $\{2\star \mid 1\}$ while $P_1(k-i-1)$ has value 1 or $1\star$. Therefore,

$$\begin{aligned} P_2(i) + P_1(k-i-1) &= (\{2 \mid 1\star\} \text{ or } \{2\star \mid 1\}) + (1 \text{ or } 1\star) \\ &= \{3 \mid 2\star\} \text{ or } \{3\star \mid 2\} \\ &> 1 \\ &= P_1(k) \end{aligned}$$

and we find that Right's option to $P_1(k)$ dominates all others.

So, for $k+1$ odd, we find that

$$P_2(k+1) = \{1 + P_1(k) \mid P_1(k)\} = \{2 \mid 1\star\}$$

And for $k+1$ even, we find that

$$P_2(k+1) = \{1 + P_1(k) \mid P_1(k)\} = \{2\star \mid 1\}$$

■

Lemma 3.2.6 *The digraph $P_3(n)$, $n \geq 5$ has value:*

$$\begin{cases} 2 & \textit{if } n \textit{ is even} \\ 2\star & \textit{if } n \textit{ is odd} \end{cases}$$

Proof: First we should note that although this class of digraphs is ultimately periodic, the first couple values do not follow the same pattern. In particular, it is easy to check that $P_3(3) = 3$ and $P_3(4) = \{3|2\star\}$. Now, for $n = 5$ we get that $P_3(5) = \{\{3 \mid 2\star\}, 2, 1\star \mid 4, 2\}$. The option to 4 for Right is dominated as well as the options to 2 or $1\star$ for Left. Therefore, we are left with $P_3(5) = \{\{3 \mid 2\star\} \mid 2\}$. Now we can see that Left's option is reversible to 2 (since $2\star \leq \{\{3 \mid 2\star\} \mid 2\}$) and we finally arrive at $P_3(5) = \{2 \mid 2\} = 2\star$.

[As an aside, it is worthwhile to point out something rather curious that just happened during the course of simplification: Left's option of 2 was dominated by his option to $\{3 \mid 2\star\}$ which means that Left should never play to 2 since it is an inferior move. Then we found that $\{3 \mid 2\star\}$ was reversed out and replaced by an option to 2. It is interesting because this means the option that was always better than moving to 2 is equivalent to moving to 2 in this particular game.]

Now that we have a base case, we can proceed with the induction by assuming the pattern will now hold up to $P_3(k)$. From the game $P_3(k+1)$ we see that Left has three legal move which are removing either end vertex with in-degree 0 or the vertex with in-degree 2. These moves give options with values $P_3(k), P_1(k)$ and $1+P_1(k-1)$ respectively. Rights options are to remove any of the other vertices leaving a graph with two components with value $P_3(i)+P_1(k-i-1)$ where $3 \leq i \leq k-1$. Therefore we get:

$$P_3(k+1) = \{P_3(k), P_1(k), 1+P_1(k-1) \mid 1\star + P_1(k-3), P_3(i)+P_1(k-i-1)\}$$

Again, we examine the options available to both players and eliminate those that are dominated to arrive at:

$$P_3(k+1) = \{P_3(k), 1+P_1(k-1) \mid 1\star + P_1(k-3)\}$$

Now, for $k \geq 6$ we find that Left's options have the same value and Right's option is again the same value. So, for $k+1$ odd, we get that

$$P_3(k+1) = \{2, 2 \mid 1+1\} = \{2 \mid 2\} = 2\star$$

Likewise, when $k+1$ is even we end up with

$$P_3(k+1) = \{2\star, 2\star \mid 1\star + 1\} = \{2\star \mid 2\star\} = 2$$

∎

Definition 3.2.2 *Let $AP_1(n)$, $AP_2(n)$ and $AP_3(n)$ be directed paths with n vertices such that each edge along the path is oriented in an alternating fashion. $AP_1(n)$ is such that both end vertices have in-degree 0, $AP_2(n)$ is such that exactly 1 of the end vertices have in-degree 0 and $AP_3(n)$ is such that both end vertices have in-degree 1.*

Due to parity, we can deduce that $AP_1(n)$ and $AP_3(n)$ must always have an odd number of vertices while $AP_2(n)$ must always have an even number of vertices.

Lemma 3.2.7 *The digraph $AP_1(n)$ has value n for all n.*

Proof: First we note that Right has no legal moves since all vertices will have in-degree 0 or 2. On the other hand, Left can remove a vertex of in degree 2 to produce a disconnected graph where the components are $AP_1(i)$ and $AP_1(j)$ where $i+j = n-1$. By induction we can see that this option will have value $n-1$. There cannot be any position on $n-1$ vertices with value greater than $n-1$, so this must be his best move. Therefore, $AP_1(n)\,\{n-1 \mid \} = n$, as desired. ∎

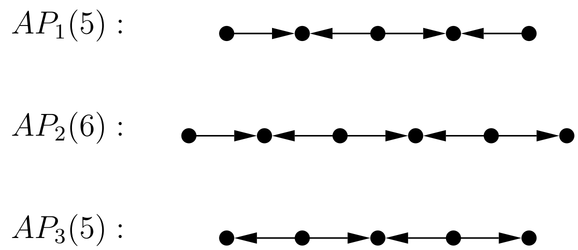


Figure 3.6: The digraphs $AP_1(5)$, $AP_2(6)$ and $AP_3(5)$

Lemma 3.2.8 *The digraph $AP_2(n)$ has value $n-2$ for all $n \geq 4$.*

Proof: We begin by checking that $AP_2(2) = 1\star$ and $AP_2(4) = 2$. There is only 1 vertex of in-degree 1, so Right only has one legal move from $AP_2(n)$ to $AP_1(n-1)$. Left can remove any of the other vertices in the digraph since they all have in-degree 0 or 2. If he removes a vertex of in-degree 2, then he leaves two components $AP_1(i)$ and $AP_2(j)$ where $i + j = n - 1$. By induction and the previous lemma we see that this option is at most $(i + j - 1)\star = (n - 2)\star$. Therefore, $AP_2(n) = \{(n-2)\star \mid n-1\} = n-2$. (Thus, the difference of these games is a second player win: $\{(n-2)\star \mid n-1\} - (n-2)$. If Left moves to $(n-2)\star - (n-2) = \star$, Right moves to 0 and wins. Likewise, if Right plays first she must move to $(n-1) - (n-2) = 1$ which is a Left win.) ∎

Lemma 3.2.9 *The digraph $AP_3(n)$ has value $n-4$ for all $n \geq 7$.*

Proof: We check the first couple values of $AP_3(n)$ to find that $AP_3(1) = 1$, $AP_3(3) = \{2 \mid 1\star\}$, $AP_3(5) = 2\star$ and $AP_3(7) = 3$. First we note that although Right has two different vertices she could remove, they both result in the digraph $AP_2(n-1)$ which we know has value $n-3$. Left can move to two different types of positions:

- $AP_3(i) + AP_3(j)$ where $i + j = n - 1$ and both i and j are odd.
- $AP_2(i) + AP_2(j)$ where $i + j = n - 1$ and both i and j are even.

In the first case, by induction we find that $AP_3(i) + AP_3(j)$ is at most $n - 5$ when we choose $i = 1$. In the second case, if we let $i = 2$ the previous lemma tells us that this option has value $1\star + (n-3) - 1 = (n-4)\star$. For any other value of i we get an option with value $(i-2)+(j-2) = i+j-4 = n-5$. Therefore, Left's best option is when he lets $i = 2$ and we get that $AP_3(n) = \{(n-4)\star \mid n-3\} = n-4$ as desired. Again, this last equality can be seen by playing the difference of the two games. ∎

The most important difference between this version of the game and it's counterpart played on undirected graphs is that there are digraphs which the Right player can win by playing first or second. The reason for this is that the sum of in-degrees is equal to the number of arcs in the digraph. Previously, we were able to use the fact that the sum of degrees in a simple graph is twice the number of edges. For example, Figure 3.7 shows a digraph D which has value -1.

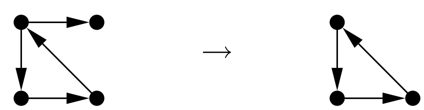


Figure 3.7: A digraph with a negative game value, -1

Since each vertex has odd in-degree, Left cannot make a legal move and would lose if he played first. If Right plays first, she may remove the vertex with 0 out-degree, as shown, leaving a digraph that again has only vertices with odd in-degree and thus winning the game. It is easily checked that this option is Right's best move and that the option has value 0. Therefore, $D = \{\ \mid 0\} = -1$.

Also, the digraph game values don't fall into the classes of even and odd games previously defined. For example, Figure 3.8 shows two different digraphs on 3 vertices with 3 arcs which have values 0 and 1 respectively.

Figure 3.8: Two 3-vertex digraphs with values 0 and 1

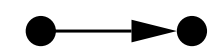
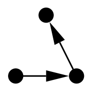
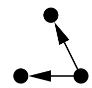
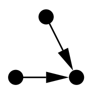
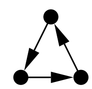
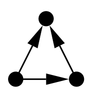
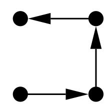
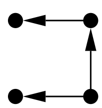
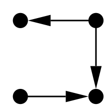
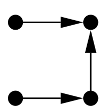
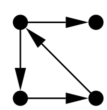
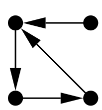
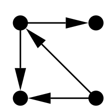
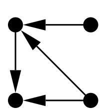
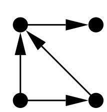
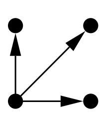
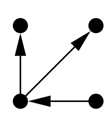
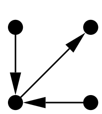
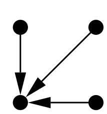
Graph	E/O	E/E	O/O	Graph	E/O	E/E	O/O
	1	$\star$	0		$1\star$	0	$\star$
	1	$\star$	$\star 2$		$\{2 \mid 1\star\}$	$\star$	0
	3	$\star$	0		0	0	0
	1	$\star$	0		$1\star$	0	0
	$\{2\star \mid 1\}$	0	$\star$		2	0	$\star$
	$\{3 \mid 2\star\}$	0	0		-1	0	$\star 3$
	$\{2\star \mid 1\}$	$\star 2$	$\star$		$1\star$	0	$\star 2$
	4	0	0		$\{2\star \mid 1\}$	0	$\star 3$
	$\{3 \mid\mid 2 \mid 1\star\}$	0	$\star$		$1\star$	0	$\star$
	$3\star$	0	$\star$		$3\star$	0	$\star$

Table 3.3: Digraph values for Even/Odd, Even/Even and Odd/Odd

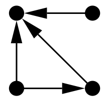
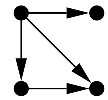
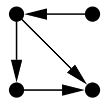
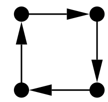
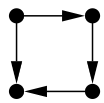
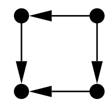
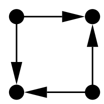
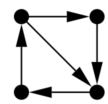
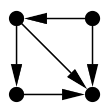
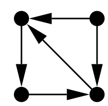
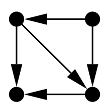
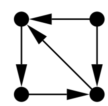
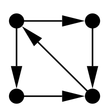
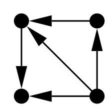
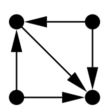
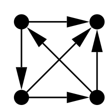
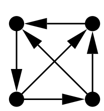
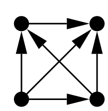
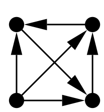
Graph	E/O	E/E	O/O	Graph	E/O	E/E	O/O
	$\{3\|2\star\}$	0	$\star 2$		$\{2\star\|1\}$	0	$\star$
	$1\star$	0	0		0	0	0
	2	0	$\star$		$\{3\|1\}$	0	0
	4	0	0		$\frac{1}{2} \pm \frac{1}{2}$	0	$\star$
	$1\star$	0	$\star$		$\{3\|1\}$	$\star 2$	$\star$
	$1\star$	0	0		$\{3\|1\}$	0	$\star$
	$\{2\|1\star\|\|0\}$	0	$\star$		2	0	$\star$
	$3\star$	0	$\star$		-1	0	0
	$\frac{1}{2} \pm \frac{1}{2}$	$\star 2$	0		$1\star$	0	0
	2	$\star 2$	0				

Table 3.4: Digraph values for Even/Odd, Even/Even and Odd/Odd (cont.)

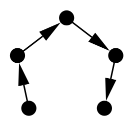
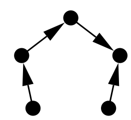
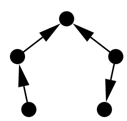
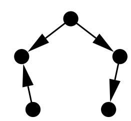
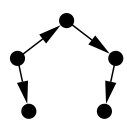
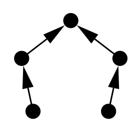
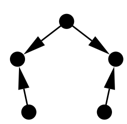
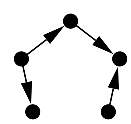
Graph	E/O	E/E	O/O	Graph	E/O	E/E	O/O
	1	⋆	⋆		2⋆	⋆	⋆
	3	⋆	⋆		2⋆	⋆	⋆2
	{2|1⋆}	⋆			2⋆	⋆	⋆2
	5	⋆	0		3 ± 1	⋆	⋆

Table 3.5: Digraph values for Even/Odd, Even/Even and Odd/Odd (cont.)

Chapter 4

Grand Left/Right

4.1 Playing The Game

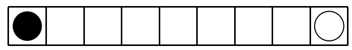
During the course of play, Grand Left/Right usually becomes the sum of small disconnected components. Also, in the smallest components, to secure a position that is an integer, empty spaces must be completely surrounded by your own pieces. Therefore, an intuitive strategy for the game might be to play such that your pieces are adjacent to as many empty spaces as possible. In that way, even if you cannot secure any empty squares for yourself, at least your opponent can no longer secure any of the spaces your piece is next to either.

For many small games, this strategy usually causes the game to break down into many small positions which all have value 0 or $\star$. Therefore, given a symmetric board and starting position, the game will be won based on parity (all spaces of the board will be filled with neither player securing any beneficial positions).

A list of all values that Grand Left/Right can take when there is a game with at most 4 free spaces is given in Table 4.1.

4.2 One Empty Row

The next two results deal with a game in which there is a single empty row left in the game. We assume that the row is completely surrounded by pieces (ie it is not at the side of the board). In the following diagram, we specify that the leftmost piece is black and the rightmost piece is white, but do not specify which pieces are above or below the empty spaces in the diagram.



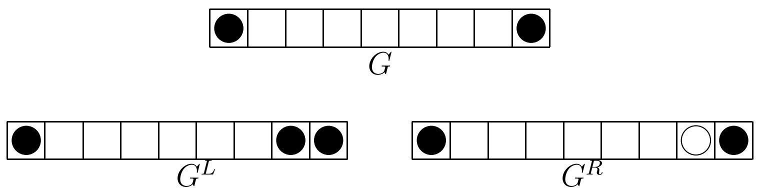
Lemma 4.2.1 *In the above position, if there are $2k$ empty spaces along a row, Left's option is $\geq \star$. If there are $2k+1$ empty spaces, Left's option is ≥ 0.*

Proof: Left can always begin by shooting across the row leaving one of his pieces at the other end. If Right has a move, it will be to a space at an end of the row, creating a similar position with exactly two fewer empty spaces. Of course, if Right has no move at some point while there are still empty spaces, then this position must have a positive value. By induction, Left can continue in this manner as long as Right does not fill the last space and create a game of value 0. In this case, we know there must be an even number of spaces left and that Left never has an option to a positive number. Therefore, his best option is to $\star$ (since Right moves to 0). Otherwise, Left wins which implies his Left option is ≥ 0. ■

Lemma 4.2.2 *If Right's best move when playing in a game with a single empty row is to a game with a positive value, then Left must have an option which is ≥ 0.*

Proof: Case 1: After Right's move, both endpoints of the path are her own pieces. Assume she loses going first in this new game. Then we know that since she can move after each of Left's moves, there must be an even number of spaces remaining. Now, if she goes second, she will still always be able to move after Left moves, and there will always be an odd number of spaces left so she must eventually make the last move and win. Therefore, this game cannot have a positive value.

Case 2: After Right's move, the one endpoint is Left's piece and the other is Right's. If this game has a positive value, then consider the position where Left had played first and moved to the same space Right had moved to as shown here:



Then we can see that Left's options from G^R are a subset of his options from G^L. Also, Right's options from G^L are a subset of her options from G^R. Therefore, it must be the case that $G^L \geq G^R > 0$. ∎

Theorem 4.2.3 *All Grand Left/Right paths with* $2n$ *spaces are even games and paths with* $2n + 1$ *spaces are odd games.*

Proof: Any game with 0 spaces has value 0, so this is an even game as required. If there is one space free, the game has value 1, -1 or $\star$ which are all odd games. Assume that all paths up to $2k$ spaces are even or odd games as appropriate. Now, a game with $2k+1$ spaces has only even games as options. By the previous theorem we know that if Right has an option to a game with positive value, so does Left. Therefore, we can never have this game be of value 0. Therefore, all the paths with $2k + 1$ spaces are odd games. Now we find that all games with $2k + 2$ spaces only have odd games as options, therefore they must be even games. By induction all paths with $2n$ spaces are even and all paths with $2n + 1$ are odd. ∎

Position	Value	Position	Value	Position	Value
●●● ● ●	1	●○● ● ●	$\star$	●○● ○ ○	-1
●●●● ● ●	2	●●○● ● ●	$\{1 \mid \star\}$	●○●● ● ○	0
●●○● ● ○	± 1	●○○● ● ○	$\{\star \mid -1\}$	●○○● ○ ○	-2
●●●●● ● ●	3	●●●●● ○ ●	$\{2 \mid 0\}$	●●○●● ● ●	1
●●●○● ● ●	$\{2 \mid \{1 \mid \star\}\}$	●○●●● ○ ●	$\{2 \mid \{\star \mid -1\}\}$	●●○●● ○ ●	$\star$
●○○●● ○ ●	$\{\{1 \mid \star\} \mid -2\}$	●●●○● ○ ○	-1	●○○○● ○ ●	$\{0 \mid -2\}$
●●○○● ○ ○	$\{\{\star \mid -1\} \mid -2\}$	●○○○● ○ ○	-3	○ ●	± 2

Position	Value	Position	Value
●●●●●● ● ●	4	●●●●●● ○ ●	$\{3 \mid \star\}$
●●○●●● ● ●	$\{1 \mid \star\}$	●●●○●● ● ●	2
●●●●○● ● ●	$\{3 \mid \{2 \mid 0\}\}$	●●○●●● ○ ●	0
●●●○●● ○ ●	± 1	●●○○●● ○ ●	$\{\star \mid -1\}$
●○●●○● ● ○	$\{\{2 \mid 0\} \mid -1\}$	●●○○●● ● ○	$\{1 \mid \{0 \mid -2\}\}$
●●●○○● ● ○	$\{\{2 \mid \{1 \mid \star\}\} \mid \{\{\star \mid -1\} \mid -2\}\}$	●○○○○● ● ○	$\{\star \mid -3\}$
●○●○○● ○ ○	-2	●●○○○● ○ ○	$\{\{0 \mid -2\} \mid -3\}$
● ○	± 3		

Table 4.1: All Grand Left/Right values with up to 4 free spaces

Chapter 5

Cookie Cutter

Lemma 5.0.4 *For any position of the game with n blocks, the $\mathcal{G}$-value of the game is at most n.*

Proof: We will proceed by induction. Clearly a game with no blocks has value 0 since there are no legal moves. A game with only one block has a single legal move to 0 and thus has value $\star$. In other words, it has $\mathcal{G}$-value 1.

Now assume the value of all games with $n-1$ or fewer blocks have $\mathcal{G}$-value of $n-1$ or less. Since on each turn we must remove at least 1 block, the options of a game with n blocks are all at most $n-1$ by our induction hypothesis. Therefore, any position with n blocks has $\mathcal{G}$-value at most n by the mex-rule. ∎

5.1 One Row Cookie Cutter

Definition 5.1.1 *An **arbitrarily large** cookie cutter is one of a fixed size that is big enough to cover all blocks remaining in the game.*

Corollary 5.1.1 *For arbitrarily large cookie cutters, the $\mathcal{G}$-value of a game with n consecutive blocks in one row is n.*

Proof: With an arbitrarily large cookie cutter, it is always possible remove any number of the remaining blocks if they are in a row. Therefore, this is equivalent to a game of Nim with one pile of n beans which has $\mathcal{G}$-value n. ∎

Lemma 5.1.2 *The game starting with a* $1 \times n$ *grid and a cookie cutter of size* k *is equivalent to the octal game* $\mathbf{0.33}\ldots\mathbf{37}$ *where there are* $k-1$ *threes in the expression.*

Proof: We begin by thinking of the n blocks as a pile of beans as in Nim. On each turn, a player may remove from 1 to $k-1$ blocks as long as it is from an end thus leaving the equivalent of 0 or 1 piles of beans remaining. This accounts for the $k-1$ threes in the expression. If a player removes k blocks he may leave 0, 1 or 2 piles since he could place the cookie cutter so as to split the initial row into two. These two piles are disjoint because the blocks are too far apart for a cookie cutter to be placed which would cover blocks from both piles. This accounts for the 7 in the k^{th} place of the expression. Finally, there is no way to remove more than k blocks since they lie in a row and the cookie cutter is of size k. Therefore, these games are equivalent. ∎

Lemma 5.1.3 *The expressions* $i + j$ *and* $i \oplus j$ *have the same parity.*

Proof: If exactly one of i and j are odd, then their sum is odd. Also, only odd numbers have a 1 in the units column of their binary representation. Therefore, the nim-sum will also have a 1 in that place making $i \oplus j$ odd. Likewise, if i and j are either both odd or both even their sum will be even. Also, there will be an even number of 1s in the units column of the binary representation, making the nim-sum even as well. ∎

Theorem 5.1.4 *When using a cookie cutter of size* k *where* k *is odd, the* $\mathcal{G}$*-sequence produced by the corresponding octal game is periodic. Moreover, the sequence has no irregular values, and* $\mathcal{G}(n) = n \ (mod(k+1))$.

Proof: The first $k+1$ terms of the $\mathcal{G}$ sequence are the numbers 0 through k in order since the cookie cutter is large enough to remove all blocks. Therefore, this is equivalent to a single pile of beans in nim.

Assume that the specified pattern holds until we get to some term $t \geq k+1$. We would like to show that this term has value $t \pmod{(k+1)}$ since that would be the next term in the sequence described. We know that we can remove up to k blocks and leave the remainder in one pile, so this gives us the options $\{t-1, t-2, ...1, 0, k, k-1...t+1\}$ (all terms $\text{mod}(k+1)$). Also, we could remove exactly k and split the remainder into two piles.

Since $k+1$ is even, then t will have the same parity as $t \pmod{(k+1)}$. When we remove k and split into two piles, of size i and j, we know that $i+j$ will have opposite parity to t. But then by the Lemma 5.1.3 we know that $\mathcal{G}(i)+\mathcal{G}(j)$ will also have opposite parity to t. In particular, this cannot be an option which is equivalent to $t \pmod{(k+1)}$. Therefore, we know that when we have a pile of size t, we have options to 0 through $t-1 \pmod{(k+1)}$, but no option to $t \pmod{(k+1)}$. By the mex-rule we conclude that $\mathcal{G}(t) = t \pmod{(k+1)}$. Therefore, by induction we have shown that the $\mathcal{G}$-sequence will always be periodic with the values specified above. ■

I have checked up to $k = 32$ and have found that although for all k odd we have periodic $\mathcal{G}$-sequences, not all of the others appear chaotic in the sense that they do not seem to be a period. In particular, for k=4, 12 and 28 we get periodic sequences but with some irregular values first.

Theorem 5.1.5 *For $k = 4(2^a - 1), a > 0$ the $\mathcal{G}$-sequence of the octal game* **0.33...37** *with $k-1$ 3s is periodic.*

Proof: In particular, we will show that the $\mathcal{G}$-sequence becomes periodic starting at $n = 4k+4$. The irregular values will be grouped into 5 sections which we will deal with in order:

- For $0 \leq n \leq k$ we find that $\mathcal{G}(n) = n$.
- For $k+1 \leq n \leq 2k+2$ we find that $\mathcal{G}(n) = n-(k+1)$.

- For $2k+3 \leq n \leq 3k+3$ we find that $\mathcal{G}(n) = n - (2k+2)$.
- For $3k+4 \leq n \leq 4k+2$ we find that $\mathcal{G}(n) = n - (3k+3)$.
- For $n = 4k+3$ we find that $\mathcal{G}(n) = 0$.

Starting at $n = 4k+4$ we observe that a period of length $k+1$ begins taking on the values $(k+1), 1, 2, ..., k$ in that order.

For the first section, we have a cookie cutter of size k, so it can remove all blocks in play if a player chooses. Therefore, we may treat this as an arbitrarily large cookie cutter and apply Corollary 5.1.1 to find that $\mathcal{G}(n) = n$ when $0 \leq n \leq k$.

For the second section, we know by induction that from $n = k+1+i$, $1 \leq ileqk+1$ we can get to any of the previous k values in the $\mathcal{G}$-sequence which are $\{i-1, i-2, ...1, 0, k, k-1, ...i+1\}$. Also, we may split the pile after removing k. If we choose to split the pile we know the two resultant piles will sum to $i+1$ since we removed exactly k from the $n = k+1+i$ that we started with. Then by Lemma 5.1.3 we know that we cannot have two piles which have a sum of $i+1$ and a nim-sum of i since they have opposite parity. Therefore, from a position with $n = k+1+i$, we have no option to a game with $\mathcal{G}$-value i but we do have all options up to $i-i$. Thus, $\mathcal{G}(n) = n - (k+1)$ for $k+1 \leq n \leq 2k+2$.

For the third section, we know by induction that from $n = 2k+2+i$ we can get to any of the previous k values in the $\mathcal{G}$-sequence which are $\{i-1, i-2, ...1, k+1, k, ...i+1\}$. Again, we may also split the pile after removing exactly k. In order to show $\mathcal{G}(2k+2+i) = i$ we still need to show that there exists an option to 0 and there is no option to i. To show there is an option to 0, we consider two cases:

- i is even: Then we can remove exactly k (which is also even to have $k+2+i$ remaining. If we split this into two equal sized piles, then each pile will have the same $\mathcal{G}$-value and their nim-sum will therefore be 0.
- i is odd: In this case, we split the pile of size $k+2+i$ into piles of size j and $k+2+i-j$. As long as $k+2+i-j$ falls into the second range described above,

it will have $\mathcal{G}$-value $i+1-j$. Since we want the values to be equal (and hence their nim-sum 0) we solve $j = i+1-j$, so $j = \frac{i+1}{2}$. This indeed puts the pile of size j into the first range and the pile of $k+2+i-j$ in the second, so this position has $\mathcal{G}$-value 0.

Now we need to show there is no option to a position with $\mathcal{G}$-value i. As above we will find that we are splitting the pile into two such that the sum of the two piles is $k+2+i$ and their nim-sum is i. Since one of the piles must fall into the second range, we know that the $\mathcal{G}$-value the piles will be j and $i+1-j$. Again, by Lemma 5.1.3, we know this can never have a nim-sum of i since the sum of $\mathcal{G}$-values is $i+1$. Therefore, from a position $n = 2k+2+i$, $1 \leq i \leq k+1$ we know we can get to all the $\mathcal{G}$-values from 0 to $i-1$ but never to i. Thus, by the mex-rule, we know that $\mathcal{G}(n) = n-(2k+2)$ for $2k+3 \leq n \leq 3k+3$.

(Note that if k is not of the form $4(2^a - 1), a > 0$ it is possible to have both piles in this third section fall into the second section after splitting and create higher $\mathcal{G}$-values.)

For the fourth section, we again need to show that we have all the options up to $i-1$ but no option to i when we start with $n = 3k+3+i$. The options from 1 to $i-1$ are obtained exactly as before and the option to 0 occurs when you remove exactly k and split into two heaps, one of which is of size $\frac{i+1}{2}$ when i is even or into equal piles when i is odd. Using a similar argument as above, we can show the option to i cannot exist. Therefore, by the mex-rule, $\mathcal{G}(n) = n-(3k+3)$ when $3k+4 \leq n \leq 4k+2$.

For $n = 4k+3$ we get all the same options as we would in section 4 except that our only option to 0 no longer exists since creating the pile of size $\frac{i+1}{2}$ no longer falls into the first range and hence the nim-sum is not 0. Since we have no option to 0 in this case, $\mathcal{G}(4k+3) = 0$ by the mex-rule.

Starting at 4k+4, we are now in the section that will repeat forever. To show why, we must examine the game starting at $n = 4k+4+i$ to show that all options from 0 to $i-1$ exist but that i does not (modulo (k+1) since that is the length of the

Figure 5.1: The first 30000 values generated by the octal game **0.33333333333337**

period). As before, showing the options from 0 to $i-1$ exist is based on induction. Showing i does not exist is again due to parity. The only case we need to be careful of is $n = 5k+4$ because we can no longer reach a position with value 0 unless we split the heap. Of course, once we have removed k, we have an even number left over so we can split into equal piles to get the $\mathcal{G}$-value 0. For all $n = 5k+4+i, i > 0$ we can show that $\mathcal{G}(n) = i$ (modulo $k+1$) by showing all the options from 0 to $i-1$ exist and an option to i does not. Therefore, this is a periodic sequence with a period length of $k+1$. ■

At the time of writing, there are no other known games of this form that become periodic. All other values of $k \leq 32$ are 'badly behaved' but generate some rather interesting looking graphs. Figure 5.1 shows the first 30000 $\mathcal{G}$-values of the game played when $k = 14$.

5.2 Two Row Cookie Cutter

Definition 5.2.1 *A position of **Type-A**, denoted **A(a,b)** has two adjacent rows with a and b consecutive blocks respectively. These rows start in the same column.*

Definition 5.2.2 *A position of **Type-B**, denoted **B(n,m)**, has 2 rows which are divided into three sections. The first (leftmost) section contains n consecutive blocks in the first row. The second section contains no blocks but as many empty columns as desired. The third (rightmost) section contains m consecutive blocks in the second row.*

Figure 5.2: Cookie Cutter positions $A(2,3)$ and $A(1,4)$

Figure 5.3: Cookie Cutter positions $B(2,3)$ and $B(1,4)$

Lemma 5.2.1 *For arbitrarily large cookie cutters, the position B(n,m) has $\mathcal{G}$-value* $n + m$

Proof: We will proceed by induction. Since $B(n, 0)$ and $B(0, m)$ are single rows, by Corollary 5.1.1 we know they have $\mathcal{G}$-values n and m respectively. Assume the statement is true for all positions for $a + b < n + m$ and $a \leq n, b \leq m$. Consider the following two types of legal moves:

- Take k blocks from the group of n leaving a position $B(n - k, m)$ where $0 < k \leq n$.

- Take all blocks from the group of n and k blocks from the group of m leaving a position $B(0, m - k)$ where $0 < k \leq m$.

By the induction hypothesis, the second move generates positions which have values 0 to $m-1$. The first move generates positions which have values from m to $m+n-1$. Therefore, by the mex-rule we know that $B(n,m)$ has $\mathcal{G}$-value at least $n+m$. By Lemma 5.0.4 we know that it can have value at most $n+m$. Thus, it's value must be exactly $n+m$. ■

Theorem 5.2.2 *For arbitrarily large cookie cutters, the position $A(a,b)$ has $\mathcal{G}$-value $a-1$ when $a=b$ and $a \neq 2^k-1$ for any $k>0$. Otherwise, $A(a,b)$ has $\mathcal{G}$-value $a+b$.*

Proof: We will proceed by induction. From Lemma 5.0.4 we know the position $A(a,0)$ has $\mathcal{G}$-value a. It is also easy to see that $A(1,1)$ is the same as $A(2,0)$ (by rotation) and therefore has $\mathcal{G}$-value 2.

Assume that the statement is true for all positions for $a+b<n+m$ and $a \leq n, b \leq m$. First we check the case when $n<m$. We want to show that we have options which take on all $\mathcal{G}$-values from 0 to $n+m-1$. Consider the following four types of moves and the $\mathcal{G}$-values they produce based on the induction hypothesis:

1. Remove blocks from the first row to reach positions $A(k,m)$ where $0 \leq k < n$ with $\mathcal{G}$-values from m to $n+m-1$.
2. Remove blocks from the second row to reach positions $A(n,k)$ where $0 \leq k < m$ with $\mathcal{G}$-values n to $2n-1$, $2n+1$ to $n+m-1$ and $2n$ if $n=2^j-1$ for some j and $n-1$ otherwise.
3. Remove blocks from both rows from the unequal side to reach positions $A(k,k)$ where $0 \leq k < n$ which have $\mathcal{G}$-values $2k$ when $k=2^j-1$ for some j and $k-1$ otherwise.
4. Remove at least n blocks from the second row starting at the equal side to reach positions $B(n,k)$ where $0 \leq k \leq m-n$ with $\mathcal{G}$-value $n+k$.

The move of type 2 generates all the values from n to $n+m-1$ except possibly $2n$.

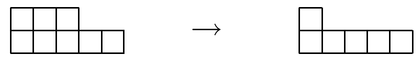


Figure 5.4: Cookie Cutter move of type 1 from $A(3,5)$ to $A(1,5)$

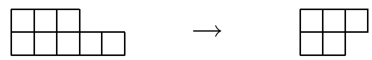


Figure 5.5: Cookie Cutter move of type 2 from $A(3,5)$ to $A(3,2)$

If $2n \geq m$ then let $m = n + k$. We can make a move of type 1 to the position $A(n-k, m)$ since $n \geq k$. But then we find that $A(n-k, m) = A(n-k, n+k)$ which has $\mathcal{G}$-value $2n$ since $k > 0$.

If $2n < m$ then we can make a move of type 4 to take $m - n$ blocks leaving the position $B(n, n)$ (since $m - n > n$). By Lemma 5.2.1, this position has $\mathcal{G}$-value $2n$.

So far, we have found options from $A(n, m)$ to positions of $\mathcal{G}$-values n to $n + m - 1$. We now need to find options with $\mathcal{G}$-values 0 to $n - 1$.

By using moves of type 3, we can generate all the positions $A(k, k)$ where $0 \leq k \leq n$. So, based on the induction hypothesis, we need to determine if all the $\mathcal{G}$-values from 0 to n-1 exist as games of this form. The $\mathcal{G}$-value $j - 1$ exists in the position $A(j, j)$ when $j \neq 2^i - 1$ for any i. If $j = 2^i - 1$ then the value $j - 1$ exists in the position $A(\frac{j-1}{2}, \frac{j-1}{2})$ since we know that $\frac{j-1}{2} = 2^{i-1} - 1$ and thus has $\mathcal{G}$-value $2(\frac{j-1}{2}) = j - 1$. Therefore, for any particular value $j \leq n - 1$, there exists a k with $0 \leq k \leq n$ such that the position $A(k, k)$ has a $\mathcal{G}$-value of j. It is now easy to check that the $\mathcal{G}$-values of these positions is exactly the set $\{0, 1, \ldots, n-1, 2j\}$ where $j = \max\{i : i \leq n$ and

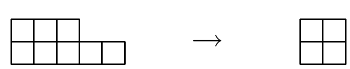


Figure 5.6: Cookie Cutter move of type 3 from $A(3,5)$ to $A(2,2)$

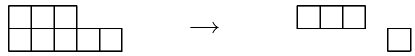


Figure 5.7: Cookie Cutter move of type 4 from $A(3,5)$ to $B(3,1)$

$i = 2^p - 1$ for some $p\}$.

In summary, we've found moves to positions which can take on all the values from 0 to $n+m-1$. Therefore, the mex-rule tells us the $\mathcal{G}$-value must be at least $n+m$. By Lemma 5.0.4 we know this is also an upper bound and therefore is exactly the $\mathcal{G}$-value of this position.

Now we assume $n = m$. From this position, there are only two distinct types of moves.

R_1 - Remove blocks from both rows leaving position $A(k,k)$ where $0 \le k \le n-1$.

R_2 - Remove k blocks from one row leaving position $A(n-k,n)$ where $0 \le k \le n-1$.

As before, we note that the moves of the type R_1 generate the set of $\mathcal{G}$-values 0,1,...,n-2, 2j where $j = \max\{i : i \le n-1$ and $i = 2^p - 1$ for some $p\}$. If $n = 2^k - 1$ for some k, then $j = \frac{n-1}{2}$ and therefore $2j = n-1$. If $n \neq 2^k - 1$ for any k, then $2j > n-1$. In particular, the value $n-1$ is only an option from this position using these moves when n is 1 less than a power of 2.

Moves of type R_2 generate positions $A(n-k,n)$ where $0 < k \le n$. Since $n-k \neq n$, then by our previous result we know this position has a $\mathcal{G}$-value of $2n-k$. Therefore, this gives all $\mathcal{G}$-values in the range n to $2n-1$.

We have now considered all possible moves from the position $A(n,n)$. When $n = 2^k - 1$ for some k, we have legal moves to positions which have all $\mathcal{G}$-values from 0 to $2n-1$. The $\mathcal{G}$-value of $A(n,n)$ is therefore at least $2n$. It is also at most $2n$ by Lemma 5.0.4 so this is its exact $\mathcal{G}$-value.

When $n \neq 2^k - 1$ for any k, then there is no legal move to a position having $\mathcal{G}$-value $n-1$. In this case, by the mex-rule we know its $\mathcal{G}$-value is $n-1$. ■

Table 5.1 shows the $\mathcal{G}$-values for different sized grids and an arbitrarily large cookie cutter. As we've seen, the values in the first two rows are completely solved. At the time of writing, there are no other values known besides those shown here. The computer program used to calculate these values was written by Kristy Anstett [1].

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	2	1	6	3	4	5	14	7	8	9	10	11	12	13	30
3	3	6	1	2	8	4	12								
4	4	3	2	1	7	11	5								
5	5	4	8	7	1	2									
6	6	5	4	11	2	3									

Table 5.1: Nimbers generated by an $x \times y$ board and an arbitrarily large cookie cutter

5.3 Other results

In the first section we showed that a row of n blocks has $\mathcal{G}$-value n when we have an arbitrarily large cookie cutter. There are many other positions that also have $\mathcal{G}$-value n when there are n blocks. I will characterize some of them here.

Clearly, any position where we can leave any number of remaining blocks will have value n. We can see this by examining the value of the options. Since from any option with k blocks we generate we can again move to a position with any number of blocks up to $k-1$, by induction we can show that this option will have $\mathcal{G}$-value k. Therefore the options from our original position include all the nimbers from 0 to $n-1$. Thus, it must have $\mathcal{G}$-value n.

Finally, I will give a theorem which describes a set of starting positions which the second player can always win along with an explicit strategy to do so.

Theorem 5.3.1 *For a cookie cutter of size k, a $(k+1) \times n$ grid has $\mathcal{G}$-value 0 for all n.*

Proof: We will proceed by induction. If $n = 0$ there are no blocks so the game has $\mathcal{G}$-value 0.

Assume that every position with a rectangle of $(k+1) \times m$ where $m < n$ has $\mathcal{G}$-value 0. On the first player's turn, he must remove between 1 and k blocks from up to k consecutive rows. On the second player's turn, he may remove between 1 and k blocks from each of exactly the same rows such that those rows are now completely empty. We can always do this since given a number from 1 to k, we may always select another number from 1 to k such that their sum is $k+1$. There are two possible types of resulting positions. First, we may have a game of the same form with at least 1 less row and thus has $\mathcal{G}$-value 0. Otherwise, the game has two disjoint components which are also of the same form and have fewer rows. This is the disjunctive sum of two games, both of which have $\mathcal{G}$-value 0 by the induction hypothesis. This sum is again 0 which means the second player has a winning strategy and the game has a $\mathcal{G}$-value of 0. ■

When you look at the table indexed by the x and y dimensions of the starting grid, the previous theorem tells us that a row and column of 0s will occur at precisely $x = k+1$ and $y = k+1$ where the cookie cutter is of size k. This has been coined a **bounding box** of 0s.

Tables 5.2 through 5.7 give $\mathcal{G}$-values for the game which begins with an $x \times y$ grid for a particular fixed sized cookie cutter. This data is again thanks to Kristy Anstett [1].

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	2	0	1	2	3	1	2	3	4	0	3	4	2	1
2	2	1	0	2	1	3	4	1	3	4	0				
3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	1	2	0	5	1										
5	2	1	0	1	0										

Table 5.2: Nimbers generated by an $x \times y$ board and 2×2 cookie cutter

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3
2	2	1	6	0	2	1	3	0	2	4	3	0	2	1	
3	3	6	1	0	3	6	4	0	2	7					
4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	1	2	3	0	5										

Table 5.3: Nimbers generated by an $x \times y$ board and 3×3 cookie cutter

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	2	3	4	0	1	2	3	4	5	1	2	3	4	5
2	2	1	6	3	0	2	1	6	3	5	8	1			
3	3	6	1	2	0	7	6	3	4						
4	4	3	2	1	0	5									
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 5.4: Nimbers generated by an $x \times y$ board and 4×4 cookie cutter

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	2	3	4	5	0	1	2	3	4	5	0	1	2	3
2	2	1	6	3	4	0	2	1	5	8	9	0			
3	3	6	1	2	8	0	4	3	11						
4	4	3	2	1	7	0	5								
5	5	4	8	7	1	0									

Table 5.5: Nimbers generated by an $x \times y$ board and 5×5 cookie cutter

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	2	3	4	5	6	0	1	2	3	4	5	6	7	8
2	2	1	6	3	4	5	0	2	1	6	9	7			
3	3	6	1	2	8	4	0	5	3						
4	4	3	2	1	7	11	0								
5	5	4	8	7	1	2	0								
6	6	5	4	11	2	3	0								

Table 5.6: Nimbers generated by an $x \times y$ board and 6×6 cookie cutter

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	2	3	4	5	6	7	0	1	2	3	4	5	6	7
2	2	1	6	3	4	5	14	0	2	1	6	8	7	3	11
3	3	6	1	2	8	4	12	0	3						
4	4	3	2	1	7	11	5	0							
5	5	4	8	7	1	2		0							
6	6	5	4	11	2	3		0							

Table 5.7: Nimbers generated by an $x \times y$ board and 7×7 cookie cutter

Chapter 6

Future Work

6.1 Vertex Deletion

Although many results were found about this game, there were just as many new questions that could be asked. For instance, in the Even/Odd variant played on undirected graphs, we found that there were only a subset of all game values that a graph could take. Further work along these lines would include examining this subset to see if it has any other interesting algebraic properties.

Also, it would be interesting to find other classes of graphs that fall into an arithmetic periodic sequence. We saw that the graphs $P_{n,k}$ and $G_{n,k}$ both had sequences of this type. A conjecture might be as follows:

Conjecture 6.1.1 *For any graph G, consider the graph G_n where we take the graph P_n and attach it to G at a particular vertex $v \in V(G)$ by identifying v with an endpoint of P_n. Then, G_n forms an arithmetic periodic sequence.*

One of the things most surprising about the periods of the graphs we've seen was that they were all of length 3 with a saltus of 1. Also, when we examined classes of digraphs, we found that they had periods of length 2. It would be interesting to see if there are other classes of graphs which produced sequences with a different period length or saltus.

Also, many of the impartial versions of the game lead to some very interesting problems as well. Is there an upper bound on the $\mathcal{G}$-value that a graph can take? Computer searches to obtain a larger dictionary of values may be needed before reasonable conjectures can be made.

6.2 Grand Left/Right

The obvious conjecture for Grand Left/Right would be to expand on the fact that all paths are even or odd games:

Conjecture 6.2.1 *Any Grand Left/Right position with an even number of unoccupied spaces is an even game. Likewise, any position with an odd number of unoccupied spaces is an odd game.*

Future work may also include looking at variations of the game where both players turn in the same direction or when the pieces don't turn at all, stopping when they reach an obstacle of any kind.

6.3 Cookie Cutter

We've seen that the $k+1$ row and column for a fixed cookie cutter of size k produce all 0s. In all the data that has been collected by the time of writing, it seems that whenever there is a 0 in the first row, there are 0s in the same position in all other rows. We are lead to the following conjecture:

Conjecture 6.3.1 *If the game which begins with a grid of size $1 \times n$ and a cookie cutter of size k has value 0, then the game played on a grid of size $i \times n$ with a cookie cutter of size k has value 0 for all $i \geq 1$.*

If the above result were true, we would have many more bounding boxes in our table of values. Also, these bounding boxes would relate to the 0s of the $1 \times n$ game which in turn relate to the octal games mentioned in that section. Therefore, to know everything about the bounding boxes of cookie cutter, we would need to know everything about the 0s of the octal games of the form **0.33...37**. Clearly, understanding these games better would help understand cookie cutter better as well.

Another variant of this game which was not discussed would be a partizan version where each player has a different sized cookie cutter, and perhaps even ones which are not square. If we were to give Left a cookie cutter of size 1×2, Right a cookie cutter of size 2×1 and didn't allow them to rotate the cookie cutters during play we would have a game that would be similar in nature to Domineering. Of course, values would be very different because we are allowed to place cookie cutters over empty spaces.

Appendix A

Graph Theory

The following are common graph theory definitions and results. They are found in [3] and [11].

A.1 Definitions

Definition A.1.1 *A **graph** (or **simple graph**) $G = (V, E)$ is a finite set of elements V (or $V(G)$) called **vertices** and a set of two-sets of vertices, E (or $E(G)$) called **edges**. These sets are often referred to as the **vertex set** and the **edge set** of G respectively.*

Definition A.1.2 *A **subgraph** $G' = (V', E')$ of a graph $G = (V, E)$ has $V' \subseteq V$ and $E' \subseteq E$. An **induced subgraph** H is any subgraph where $\forall x, y \in V(H)$ we have that $\{x, y\} \in E(H)$ iff $\{x, y\} \in E(G)$.*

Definition A.1.3 *Vertices x and y are **adjacent** in a graph G if $\{x, y\} \in E(G)$. Also, we say the edge $\{x, y\}$ is **incident** with the vertices x and y.*

Definition A.1.4 *The **closed neighbourhood** of a vertex v is denoted $N[v]$ where $N[v] = \{v\} \cup \{x : x \text{ is adjacent to } v\}$.*

Definition A.1.5 *The **open neighbourhood** of a vertex v is denoted $N(v)$ where $N(v) = \{x : x \text{ is adjacent to } v\}$.*

Definition A.1.6 *The **degree** of a vertex v, d(v) in a graph G is the number of vertices which are adjacent to v in G.*

Definition A.1.7 *A **path** is alternating sequence of vertices and edges (starting and ending with a vertex) such that each edge is incident with the vertex that comes before and after it in the sequence. Furthermore, the sequence cannot repeat any vertex (and hence any edge).*

Definition A.1.8 *A graph is **connected** if for every pair of vertices, there exists a path that contains them. Otherwise, we call the graph **disconnected**.*

Definition A.1.9 *A **component** H of a graph G is a connected induced subgraph of G. It must also be maximal in the sense that is if $x \in V(H)$ and there is a path from x to y in G, then $y \in V(H)$.*

Definition A.1.10 *A **cycle** is a path with the exception that the first and last vertex in the sequence can (and must) be the same. A graph which contains no cycles is called **acyclic**.*

Definition A.1.11 *A **tree** is an acyclic connected graph.*

Definition A.1.12 *A **forest** is a (possibly disconnected) acyclic graph. It is easy to see that a forest is simply a union of trees.*

Definition A.1.13 *The graph K_n is the **complete graph** on n vertices. It is the graph with n vertices and all possible edges between them.*

Definition A.1.14 *The graph S_n is the **star** on n vertices. It is the graph with n vertices where one vertex is adjacent to all others and no other edges exist.*

Definition A.1.15 *The graphs P_n and C_n are the paths and cycles on n vertices respectively.*

Definition A.1.16 *A **directed graph** (or **digraph**) $D = (V, E)$ is a finite set of vertices V and a set of ordered pairs of vertices, A (or $A(D)$) called **arcs**. An arc (x,y) is said to **originate** at x and **terminate** at y.*

Definition A.1.17 *The **in-degree** of a vertex v, $d^+(v)$ in a directed graph D is the number of edges which terminate at v in D*

Definition A.1.18 *The **out-degree** of a vertex v, $d^-(v)$ in a directed graph D is the number of edges which originate at v in D*

A.2 Useful Results

Theorem A.2.1 *For any graph G,* $\sum_{v \in V(G)} d(v) = 2|E(G)|$.

Proof: The degree of a vertex v is the same as the number of edges incident with it. Therefore, every edge $\{x, y\}$ is counted twice in the above sum because x is adjacent to y and y is adjacent to x. ■

Corollary A.2.2 $\sum_{v \in V(G)} d(v)$ *is even for every graph G.*

Theorem A.2.3 *If $|V(G)|$ is odd then there is some vertex $v \in V(G)$ which has even degree.*

Proof: Assume for a contradiction that every vertex in G has odd degree. Then $\sum_{v \in V(G)} d(v)$ must again be odd since it is the sum of only odd numbers and has an odd number of terms. This contradicts Corollary A.2.2 . ■

Theorem A.2.4 *Let G be a tree with $|V(G)| = n$. Then $|E(G)| = n - 1$*

Proof: We will proceed by induction. Clearly, if there is only 1 vertex, then there are 0 edges as required. Now assume that all trees on k vertices have exactly $k - 1$ edges. Let G be a tree on $k + 1$ vertices with an edge (x, y). $G - (x, y)$ must now be disconnected since there is no path between x and y - otherwise there would have been a cycle in G. There are now two components each of which is connected and contains no cycles. Thus each component is a tree with at most k vertices. Therefore, there are exactly $k-2$ edges left by the induction hypothesis. So, G must of originally had k edges as required. ■

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