Fibonacci and Lucas Sequences

The *Fibonacci sequence* is defined by

$$F_0 = 0,\ F_1 = 1,\ \text{and } F_{n+2} = F_{n+1} + F_n,\ \text{for } n \geq 0.$$

The *Lucas sequence* is defined by

$$L_0 = 2,\ L_1 = 1,\ \text{and } L_{n+2} = L_{n+1} + L_n,\ \text{for } n \geq 0.$$

So they satisfy the same recurrence relation with different initial values.

In this project we will learn about some common properties that these sequences share because they do satisfy the same recurrence relation, but we will also learn about some remarkable differences between these sequences.

Some Easier Results. Here are first a couple of results that you can derive directly from the recurrence relation, or that can be proved by techniques we have seen before such as induction.

(1) Show that for all $n \geq 2$, $F_{n+1}^2 - F_n^2 = F_{n+2}F_{n-1}$.
(2) Show by induction on n that for all integers $n, m \geq 1$, $F_{n+m} = F_{m+1}F_n + F_mF_{n-1}$.

Polynomial Relationships. Start by creating a table containing the first 13 Fibonacci and Lucas numbers (or more, if you like). Over the years people have found many interesting polynomial relationships between these numbers. So I want to encourage you to try to find some conjectures for relationships between these numbers. For example, what would you get if you consider the products F_nL_n (where can you find these numbers in the existing sequences)? Could you find a way to write each number in the Lucas sequence as a sum of two Fibonacci numbers? Do you see other relations? Which relations can you prove already? (Don't worry if you cannot prove them all; some of them need theory that we are going to learn about in this project.)

Generating Solutions. In our course we learned how to verify a formula for the Fibonacci numbers (by induction), but we didn't learn how you would find such a formula. In this section you will learn a technique that can be extended to work for all linear recurrence relations, but for now we will just look at $u_{n+1} = u_n + u_{n-1}$ (with arbitrary initial values).

We want to find some special generating solutions v_n and w_n that satisfy the equation $u_{n+1} = u_n + u_{n-1}$, but not necessarily a specific initial condition. Any linear combination of v_n and w_n will then also satisfy the equation. So we can then take a linear combination $rv_n + sw_n$ that will satisfy the equation with the initial conditions. Our guess for solutions is that they will be of the form $u_n = x^n$. When we substitute this, we get $x^{n+2} - x^{n+1} - x^n = 0$. So we need to find x such that $x^2 - x - 1 = 0$. Call the solutions α and β. We usually don't need to calculate α and β themselves. It is often sufficient to use what we know about their sum and their product.

(1) What are $\alpha+\beta$ and $\alpha\beta$? (Note that these numbers α and β themselves are not integers, but their sum and product are!)
(2) Verify that $u_n = r\alpha^n + s\beta^n$ is a solution of the recurrence relation without the initial condition for any s and r.
(3) If $u_n = r\alpha^n + s\beta^n$ satisfies the initial conditions for the Fibonacci sequence, what can you say about $r+s$ and $r\alpha+s\beta$?
(4) If $u_n = r\alpha^n + s\beta^n$ satisfies the initial conditions for the Lucas sequence, what can you say about $r+s$ and $r\alpha+s\beta$?
(5) Give a formula for F_n and L_n in terms of α and β.
(6) Prove your formula for F_nL_n from the first section (or derive one now).
(7) What can you say about F_nL_{n-1}? Can you find any other products or sums?

Symmetric Polynomials. A polynomial in two variables $P(x,y)$ is called *symmetric* if $P(x,y) = P(y,x)$. (When you switch the two variables, you get the same polynomial.) You may use the following result:

Theorem 1. *Any symmetric polynomial with integer coefficients in x and y can be written as a polynomial with integer coefficients in $a = x+y$ and $b = xy$.*

(1) Show that this implies that every symmetric polynomial expression in α and β is an integer.
(2) Show that $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$.
(3) What is $F_{n+1}^2 + F_n^2$?
(4) The polynomial $\alpha^4 + \alpha^2\beta^2 + \beta^4$ is symmetric in α and β. According to the Theorem just stated, it is possible to express $\alpha^4 + \alpha^2\beta^2 + \beta^4$ as a polynomial in $\alpha+\beta$ and $\alpha\beta$. Can you find this expression?
(5) Show that if $m|n$ (*i.e.*, there is an integer k such that $n = mk$), then $F_m|F_n$, *i.e.*, $\frac{F_n}{F_m}$ is an integer. (Hint: You can either do this by showing that $\frac{F_n}{F_m}$ can be expressed as a symmetric polynomial in α and β, or you may want to use one of the easier results from this project.)
(6) Show that no two consecutive Fibonacci numbers have a common factor which is greater than 1.
(7) Can you also show that if $d = \gcd(m,n)$, then $\gcd(F_m, F_n) = F_d$?
(8) Show that if F_m is divisible by F_n, then m is divisible by n.
(9) Show that for the Lucas numbers, $m|n$ does not imply that $L_m|L_n$.
(10) Show that no two consecutive Lucas numbers have a common factor which is greater than 1.
(11) We say that m *divides n oddly*, if there exists an odd number k such that $n = mk$. Show that if $m|n$ oddly, then $L_m|L_n$.

Divisibility. Given an integer q, for which positive numbers n is F_n divisible by q?

(1) Show that $5|F_n$ if $5|n$. (Hint: start by considering the remainders of the Fibonacci number after division by 5 for the beginning of the sequence. How far do you need to go?)
(2) Show that there is no Lucas number L_n such that $5|L_n$.
(3) If you want to answer the question of this section for any other positive integer q, you could use the same technique as we used for 5 in the first problem, but the question is whether there will always be a 0 in the sequence of remainders. Use the pigeon hole principle to show that there always has to be a 0.

There are many interesting results about numbers that are divisors of Lucas numbers, but they will need to wait till another project (unless you really want to learn more about this; then ask me).

Some Geometry. Consider a path consisting of three consecutive straight line segments in the plane connecting the points $(0,0)$, $(5,8)$, $(8,13)$ and $(13,21)$. Place this path inside the rectangular box with vertices $(0,0)$, $(13,0)$, $(13,21)$, and $(0,21)$.

(1) Show that the area below the path is equal to the area above the path. (For a nice proof by symmetry, add some horizontal and vertical lines to your diagram to divide the areas into rectangles.)
(2) You have probably observed that the sequence of coordinates here consists of consecutive Fibonacci numbers. So we may ask ourselves: does this work for any sequence of consecutive Fibonacci numbers? Does it also work for Lucas numbers? Show that this result is true for any path connecting a sequence of points of the form $(0,0)$, $P_1(v_n, v_{n+1})$, $P_2(v_{n+1}, v_{n+2})$, ... , $P_{2k-1}(v_{n+2k-2}, v_{n+2k-1})$, where the v_i satisfy the equation $v_{n+2} = v_{n+1} + v_n$.
(3) Would it work for a sequence of the form

$$(0,0),\ P_1(v_n, v_{n+1}),\ P_2(v_{n+1}, v_{n+2}), \ldots,\ P_{2k}(v_{n+2k-1}, v_{n+2k}),$$

where the v_i satisfy the equation $v_{n+2} = v_{n+1} + v_n$?

Zeckendorf's Representations. Edouard Zeckendorf was a Belgian amateur mathematician, who lived from 1901 until 1983. He took the Fibonacci sequences with F_1 deleted:

$$1, 2, 3, 5, 8, 13, 21, \ldots$$

and showed that every number can be written in a unique way as a sum of nonconsecutive Fibonacci numbers.

(1) Find the Zeckendorff representation for 20, for 50, and for 100.
(2) Can you proof Zeckendorf's result and give an algorithm to find the representation for any positive integer?

The Zeckendorf representation can be used to quickly convert miles into kilometers and vice versa without multiplying. The trick is that $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \alpha$, where $\alpha = \frac{1}{2}(\sqrt{5}+1) \approx 1.618$, which is close to 1.609, the number of kilometers to a mile. So if you take the Zeckendorf representation of number of kilometers and then replace each number in this representation by the next Fibonacci number, you get a good approximation for the corresponding number of miles. Conversely, you get the kilometers from the miles, by taking the sum of the previous Fibonacci numbers. Do this for 80 miles and 40 kilometers.