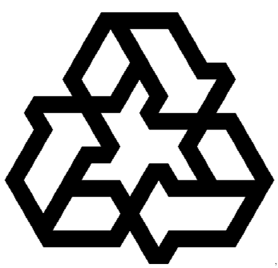
## Wallpaper Patterns

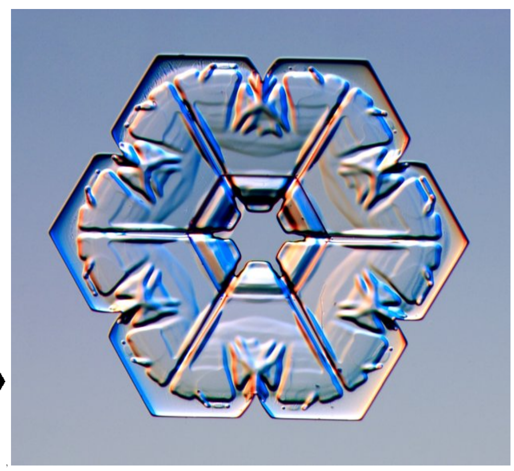
**Symmetry.** We define the symmetry of objects and patterns in terms of isometries. Since this project is about wallpaper patterns, we will only consider isometries in $\mathbb{E}^2$, and not in higher (or lower) dimenions.

**Definition 1.** An *isometry* is a function from the plane $\mathbb{E}^2$ to itself that preserves distances. So $f\colon \mathbb{E}^2 \to \mathbb{E}^2$ is an isometry if and only if for all points $P, Q \in \mathbb{E}^2$, $|PQ| = |f(P)f(Q)|$ (where the absolute value lines indicate the length of a line segment).

If $S$ is a pattern in the plane, we describe its symmetry by giving the list of isometries that leave $S$ invariant or, in other words, that map the pattern to itself.

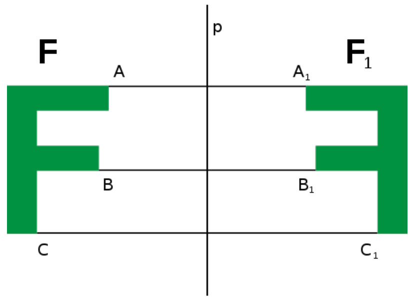
(1) Show that the composition of two isometries is again an isometry.
(2) Show also that if two isometries $\sigma_1$ and $\sigma_2$ leave a given pattern invariant, then $\sigma_1 \circ \sigma_2$ also leaves it invariant.
(3) For each of the following patterns, describe the isometries that leave it invariant:





The isometries of $\mathbb{E}^2$ are all of one of the following forms:

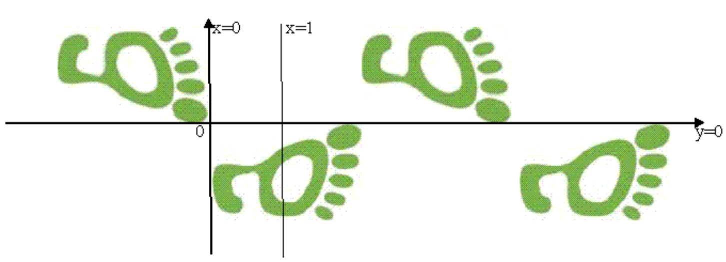
- **Reflections against lines** if you want to describe one of them, you need to give the line.




- **Rotations around points** if you want to describe one of them, you need to give the point (the center of the rotation) and the angle. Note that if the angle is $\pi$ radians, the rotation is also called a reflection across a point.


- **Translations along vectors** if you want to describe one of them, you need to give the vector (with length and direction).

- **Glide Reflections** glide reflections are obtained as a composition of a translation with a reflection, so that the reflection is against a line that is parallel to the vector of the translation. In order to describe one of them, you need to give the line, and the length of the translation vector.



It is possible to prove that every isometry of the Euclidean plane is of one of these forms.

(1) Suppose that you have two intersecting lines $l$ and $m$. Describe the isometry that you get by first performing a reflection against $l$ and then a reflection against $m$ as an isometry of one of the types listed here.
(2) Suppose that you have two parallel lines $l$ and $m$. Describe the isometry that you get by first performing a reflection against $l$ and then a reflection against $m$ as an isometry of one of the types listed here.
(3) Draw or find a pattern that has both rotational and reflection symmetry.
(4) Draw or find a pattern that has both rotational and translation symmetry.
(5) Draw or find a pattern that has both reflection and translation symmetry.
(6) Draw or find a pattern that is invariant under two translations for which the vectors are not parallel.

**Lattice Types.** Since a wallpaper pattern needs to be such that you can continue to repeat it over and over in all directions, its symmetry needs to contain two translations with vectors that are not parallel. We will start our study of wallpaper patterns by learning to find two such vectors for each wallpaper pattern.

(1) Create or find a wallpaper pattern that has only translation symmetry, no other symmetries, and give two non-parallel translation vectors for which that corresponding translations leave the pattern invariant.
(2) Could you give two different translation vectors (that are starting at the same point) for the same pattern?
(3) Consider the parallelogram that has the two translation vectors as its sides. Verify that the pattern inside the parallelogram contains all the information about the wallpaper. Suppose that you had a piece of paper with this parallelogram on it and a photocopier. How could you recreate the wallpaper pattern?
(4) Is your parallelogram minimal, or could you have done the same thing with a part of your parallelogram?

For any two non-parallel vectors $\mathbf{v}$ and $\mathbf{w}$ we can create the *lattice* of points of the form $n\mathbf{v} + m\mathbf{w}$ where $+$ stands for vector addition and $n$ and $m$ are integers.

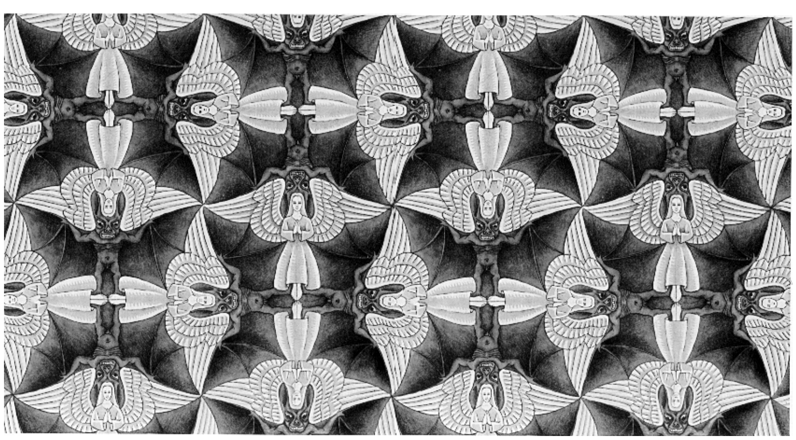
(1) Take two vectors of minimal length whose translations leave the wallpaper pattern invariant. Then draw the lattice corresponding to these vectors on your wallpaper pattern. What do you notice about these points?

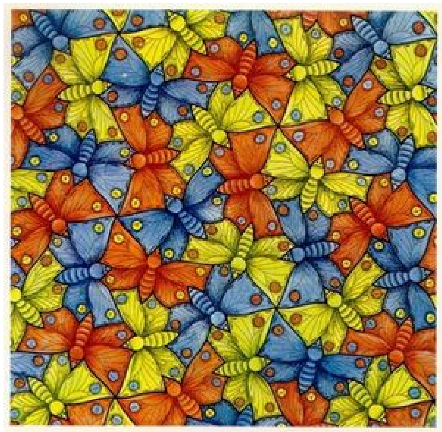
The lattice you have just created is the lattice of your wallpaper pattern. You can of course move the lattice and get a new one. However, all lattices you can make for a given pattern have the same shape. So our first way to characterize wallpaper patterns is by their *lattice type*. There are the following lattice types: hexagonal lattices, square lattices, rectangular lattices, rhombic lattices, and parallelogrammic lattices.

(1) Draw or find a picture of each type and describe its characteristics.
(2) Describe how you would go about determining of what type a given lattice is?

**The Fundamental Domain.** Once you have found the parallelogram that is spanned by the minimal translation vectors, you will notice that it can be subdivided in a number of minimal pieces that contain all the information about the wallpaper pattern with no repeats. Such a piece is called a *fundamental domain*.

(1) For the following wallpaper patterns find the lattice and a fundamental domain. (Hint: when you need to find the lattice for a wallpaper pattern with further symmetry such as rotations or reflections, use points with highest symmetry as your starting points for your vectors. It is also useful to take your vectors along reflection or glide reflection lines.)



**Which Rotations?** The main reason that we cannot have more than 17 types of wallpaper patterns is the fact that we don't have a lot of options for the angles of the rotational symmetry that we can allow in our pattern.

(1) To get a feeling for what is going on, find out which regular $n$-gons you could use to tile the plane, it i.e., which $n$-gons could be used for an edge-to-edge tiling without gaps or overlaps? You don't need to give a precise proof at this moment, just show your work and what you have learned about various values of $n$: show both why some numbers work and why others don't seem to work.

(2) The main result that restricts which rotation angles we can use is the *crystallographic restriction theorem*. You can find several proofs for this result on Wikipedia:

    http://en.wikipedia.org/wiki/Crystallographic_restriction_theorem

    Choose a proof that you like and understand and write it down in your own words.

(3) Which rotations are allowed? Give an example of each of these.

If we have a symmetry pattern involving a rotation with angle $2\pi/n$, then applying this rotation $n$ times will get us back to the identity map, so we say that that is a *rotational symmetry of order $n$*.

**Which Reflections?** Since any two reflections against lines that go through a common point compose to give a rotation, the crystallographic restriction theorem restricts also what kind of reflection symmetry we can include in a wallpaper pattern. A symmetry pattern involving $n$ reflection lines intersecting at a common point is said to give a *dihedral symmetry of order $2n$*.

(1) Give a list of the possible angles between the reflection lines going through a common point. So which dihedral symmetries occur as a part of a wallpaper symmetry.

(2) Give examples of wallpaper patterns with each of these reflection symmetries.
(3) Find some examples of wallpaper patterns that involve glide reflections. Can you make any observations about when they occur?
(4) For your favourite two wallpaper patterns give a description of the symmetry that would completely determine that particular pattern (once you have given the person the pattern on the fundamental domain).

**Websites.** Here is a list of websites that may be helpful in your research:

- www.snowcrystals.com
- http://www.mathsisfun.com/geometry/transformations.html
- http://www.mathsisfun.com/geometry/symmetry.html
- http://www.mathsisfun.com/geometry/tessellation.html
- http://www.mathsisfun.com/geometry/tessellation-artist.html
- http://www.mathsisfun.com/geometry/symmetry-artist.html
- http://www.singsurf.org/wallpaper/wallpaper.php
- http://web.inter.nl.net/hcc/Hans.Kuiper/
- http://escher.epfl.ch/escher/
- http://www.scienceu.com/geometry/handson/kali/
- http://www.clarku.edu/ djoyce/wallpaper/