Solutions to Selected Problems from the Review Sheet for the Final Exam of MATH 1600 - Fall 2009

1. Sets and Proofs

Typical Problems. Those from the assignments, but if you like to try something new, here are a couple more:

(1) Which of the following sets are subsets of each other:

$$\begin{aligned} U &= \{u \in \mathbb{Z} |\, u^3 \leq 8\}, \\ X &= \{x \in \mathbb{Z} |\, x^2 < 5\}, \\ Y &= \{y \in \mathbb{R} | |y| < 4\}, \\ Z &= \{z \in \mathbb{R} |\, z^2 = 1\}. \end{aligned}$$

Determine for each pair whether or not they are subsets.

Solution A general comment on how you prove such statements: if one set is a subset of another set, you need to give an argument showing that every element of the first set is also an element of the second set; if you want to show that a given set is not a subset of another set, you need to give an element of the first set that is not in the second set.

$U \not\subseteq X$, because $-5 \in U$ (since $(-5)^3 = -125 \leq 8$), but $-5 \notin X$, since $(-5)^2 = 25 \not< 5$.

$U \not\subseteq Y$, because $-5 \in U$, but $-5 \notin Y$.

$U \not\subseteq Z$, because $-5 \in U$, but $-5 \notin Z$.

$X \subseteq U$: it is not hard to see that all elements of X are less than or equal to 2 so for each $x \in X$ we have that $x^3 \leq 2^3 = 8$, so they are in U.

$X \subseteq Y$: it is not hard to see that $X = \{-2, -1, 0, 1, 2\}$, so for all $x \in X$ we have that $|x| \leq 2 < 4$, so they are in Y.

$X \not\subseteq Z$: $2 \in X$, but $2^2 \neq 1$, so $2 \notin Z$.

$Y \not\subseteq U$, because $0.5 \in Y$, but not in U, because U only contains integers.

$Y \not\subseteq X$, because $0.5 \in Y$, but not in X, because X only contains integers.

$Y \not\subseteq Z$, because $0.5 \in Y$, but not in Z, because $0.5^2 \neq 1$.

$Z \subseteq U$, since $Z = \{-1, 1\}$ and both $1^3 = 1 \leq 8$ and $(-1)^3 = -1 \leq 8$, so $-1, 1 \in U$.

$Z \subseteq X$, since for each $z \in Z$, $z^2 = 1 < 5$, so $z \in X$.

$Z \subseteq Y$, since for each $z \in Z$, $|z| = 1 < 4$, so $z \in Y$.

(2) Give a careful proof of the fact that if n is an integer such that n^2 is a multiple of 5, then n is a multiple of 5.

Proof: Let n be an integer such that n^2 is a multiple of 5. Suppose (toward a contradiction) that n is not a multiple of 5. Then we can write $n = 5k + r$ where r is an integer with $1 \leq r \leq 4$. And $n^2 = (5k + r)^2 = 5k^2 + 10kr + r^2$. Since n^2 is a multiple of 5 and $5k^2 + 10kr = 5(k^2 + 2kr)$ is a multiple of 5, we conclude that $r^2 = n^2 - (5k^2 + 10kr)$ is a multiple of 5. But $r \in \{1, 2, 3, 4\}$ and therefore $r^2 \in \{1, 4, 9, 16\}$ which does not contain any multiples of 5. Contradiction.

We conclude that n has to be a multiple of 5.

(3) For the following statement, form its negation, and either prove that the statement is true or prove that its negation is true: $\exists x \in \mathbb{Z}$ such that $\forall n \in \mathbb{Z}, x \neq n^2 + 2$.

Solution: The negation of this statement is $\forall x \in \mathbb{Z}$, $\exists n \in \mathbb{Z}, x = n^2 + 2$.

The original statement is true: Take $x = 1$. We will show by contradiction that $\forall n \in \mathbb{Z}, 1 \neq n^2 + 2$ If there were an $n \in \mathbb{Z}$ such that $1 = n^2 + 2$, then we would have $n^2 = -1$ and this is impossible. So $\forall n \in \mathbb{Z}, 1 \neq n^2 + 2$, and we conclude that $\exists x \in \mathbb{Z}$ such that $\forall n \in \mathbb{Z}, x \neq n^2 + 2$.

2. Complex Numbers

Typical Problems.

- Find the real and imaginary parts of powers of complex numbers such as $(1-i)^{21}$ or $(1-i\sqrt{3})^{-15}$

  Solution: Let $z = 1 - i$. Then the argument of z is $-\frac{\pi}{4}$ and $|z| = \sqrt{2}$, so $z = \sqrt{2}e^{-i\frac{\pi}{4}}$. So $z^{21} = (\sqrt{2})^{21}e^{-i\frac{\pi}{4}*21} = 2^{10}\sqrt{2}e^{i\frac{3\pi}{4}} = 1024\sqrt{2}(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}) = 1024(-1+i) = -1024 + 1024i$.

  Let $u = 1 - i\sqrt{3}$. Then $|u| = \sqrt{1+3} = 2$, and the argument of u is $-\frac{\pi}{3}$, so $u = 2e^{-i\frac{pi}{3}}$ and $u^{-15} = \frac{1}{2^15}e^{i\frac{15\pi}{3}} = \frac{1}{2^15}e^{i\pi} = -\frac{1}{2^15}$.

- For which values of n is $(\sqrt{3} - i)^n$ an imaginary number? And for which values of n is it a real number?

  Solution: Let $z = \sqrt{3} - i$. Then the argument of z is $-\frac{\pi}{6}$. So z^n is an imaginary number if and only if $-\frac{\pi}{6}n = 2k\pi \pm \frac{\pi}{2}$ for some integer k. This is the case if and only if $n = -12k \pm 3$ for some integer k, i.e. if $n \equiv \pm 3 \mod 12$.

- Find a formula for $\sin(6\theta)$ in terms of $\cos\theta$ and $\sin(\theta)$.

  Solution: By de Moivre's formula, $(\cos\theta + i\sin\theta)^6 = \cos(6\theta) + i\sin(6\theta)$. We will now give the terms from the left hand side that form its imaginary part (the ones with the odd exponents) :

$$\begin{aligned}\binom{6}{1}\cos\theta * (i\sin\theta)^5 + \binom{6}{3}(\cos\theta)^3 * (i\sin\theta)^3 + \binom{6}{5}(\cos\theta)^5 * (i\sin\theta) = \\ = \quad i6\cos\theta\sin^5\theta - i20\cos^3\theta\sin^3\theta + i6\cos^5\theta\sin\theta.\end{aligned}$$

- If you consider all the roots of the equation $z^8 = 1 + i$, what shape do they make?

Solution: They form the corners of a regular octagon which is inscribed in a circle of radius $\sqrt[16]{2}$, and one of the vertices lies on a line through the origin that makes an angle of $\frac{\pi}{32}$ with the positive x-axis.

3. Induction

Typical Problems.

- Use induction to prove result about divisibility: For all integers $n \geq 0$, the number $5^{2n} - 3^n$ is a multiple of 11.

  Solution: We prove this by induction on n.
  Induction basis: for $n = 0$, $5^{2n} - 3^n = 1 - 1 = 0$, which is a multiple of 11.
  Induction hypothesis: assume that $5^{2n} - 3^n$ is a multiple of 11, say $5^{2n} - 3^n = 11k$.
  Induction step: we show now that $5^{2n+2} - 3^{n+1}$ is a multiple of 11.

$$\begin{aligned} 5^{2n+2} - 3^{n+1} &= 5^{2n} * 5^2 - 3^n * 3 \\ &= 25 * 5^{2n} - 3 * 3^n \\ &= 22 * 5^{2n} + 3 * 5^{2n} - 3 * 3^n \\ &= 11 * 2 * 5^{2n} + 3(5^{2n} - 3^n) \\ &= 11 * 2 * 5^{2n} + 3 * 11k \end{aligned}$$

  by the induction hypothesis, and it is clear that this last expression is a multiple of 11.
  We conclude that for all integers $n \geq 0$, the number $5^{2n} - 3^n$ is a multiple of 11.

- Induction can also be very helpful in proving inequalities. For the induction step in this type of situation, you want to start with the left hand side of the equation and rewrite it until the left hand side of the induction hypothesis is part of it - then you can apply the induction hypothesis to get the first inequality; after that you may need to do a bit of rewriting with the remaining terms to show that they keep you on the correct side of the inequality. Here are some practice problems:
  (1) If $n \geq 3$ is an integer, then $5^n > 4^n + 3^n + 2^n$.

  Solution: We prove this by induction on n.
  Induction basis: for $n = 3$, $125 > 64 + 27 + 8 = 99$, so this is correct.
  Induction hypothesis: assume that $5^n > 4^n + 3^n + 2^n$.
  Induction step: we will now show that $5^{n+1} > 4^{n+1} + 3^{n+1} + 2^{n+1}$.

$$\begin{aligned} 5^{n+1} &= 5 * 5^n \\ &> 5(4^n + 3^n + 2^n) \text{ (by the induction hypothesis)} \\ &= 5 * 4^n + 5 * 3^n + 5 * 2^n \\ &> 4 * 4^n + 3 * 3^n + 2 * 2^n \\ &= 4^{n+1} + 3^{n+1} + 2^{n+1}. \end{aligned}$$

- Use induction to show results about geometric problems, such as the number of areas formed by n lines in the plane (see the assignment problems). If you want more practice, here two related problems:
  (1) Given n circles in the plane, suppose that you want to colour the finite regions formed by those circles, in such a way that if two regions share a circle segment (not just a point), then they have distinct colours. How many colours would you need? Prove your result.

  Solution: You can do this with two colours. We will prove this by induction on n, the number of circles.
  Induction basis: if you have just one circle, you can colour the inside with one colour and the outside with the other colour, so it is true for $n = 1$.
  Induction hypothesis: Suppose that you can colour the regions formed by n circles with 2 colours.
  Induction step: suppose that you have $n + 1$ circles in the plane. First remove one circle. Now you have n circles left. You can colour the regions formed by these circles with two colours according to the induction hypothesis. Now add the last circle back in and swap all the colours of the regions that are inside the last circle. You may check that this gives again a valid colouring.

4. Infinity

Typical Problems.

- Determine for each of the following sets whether they are finite (give the size), countable or uncountable (give a proof):
  (1) the set of infinite 01-sequences;

  Solution: You have shown in yor last assignment that this is uncountable (using a diagonal argument).

  (2) the set of lines through the origin in the plane;

  Solution: Mapping a line through the origin to its slope gives a bijective correspondence between this set and the real numbers. The real numbers have been shown to be uncountable, so this set is also uncountable.

  (3) $\mathbb{N} \times \mathbb{Z} = \{(n, z) | n \in \mathbb{N},\ z \in \mathbb{Z}\}$;

  Solution: This set is countable. It is clear that this set is infinite, so this can be proved by constructing an injective function $f \colon \mathbb{N} \times \mathbb{Z} \to \mathbb{N}$. I define f by

$$f(n, z) = \begin{cases} 2^n 3^z & \text{if } z \geq 0 \\ 2^n 5^{-z} & \text{if } z < 0 \end{cases}$$

  You may check that this function is indeed injective.

(4) the irrational numbers;

Solution: This set is uncountable. The proof goes by contradiction. Suppose that this set is countable. The real numbers form the union of this set together with the rationial numbers. The rational numbers are known to be countable, so this would make the real numbers countable since the union of two countable sets is countable. Contradiction, so the set of irrational numbers is uncountable.

(5) the set containing all prime numbers;

Solution: This is an infinite subset of the natural numbers, so it is countable. (Do you know how to prove that there are infinitely many prime numbers?)

(6) the set of points in the plane with coordinates (n^2, m^2), where n and m are integers;

Solution: This set is countable: it is clearly infinite and an injective function f from this set to the natural numbers can be defined by $f((n^2, m^2)) = 2^{n^2} * 3^{m^2}$. (and in many other ways)

(7) the set of infinite sequences of digits;

Solution: Since this set contains the set of 01-sequences, it is uncountably infinite.

(8) the set of finite sequences of digits;

Solution: This set is countable. Think of a way to prove this.

(9) the set of finite sequences of digits of length less than or equal to 10 (if you find 10 hard, first try 4 or 5).

Solution: This set is finite of size 10^{10}.

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