Solutions to the problems from Chapter 1:

1. The correct answers are: TTFTFFTFT.

   Explanation: (a) α is indeed listed among the elements of A; (b) α and $\{1,\alpha\}$ are elements of A, but $\{\alpha\}$ is not in the list; (c) $\{1,\alpha\}$ is an element of A, but not a subset; for it to be a subset, both α and 1 would have to be elements of A, and 1 is not an element of A; (d) both 3 and $\{3\}$ are elements of A, so $\{3,\{3\}\} \subseteq A$; (e) $\{1,3\}$ is not in the list of elements of A; (f) since $\{1,3\}$ is not an element of A, $\{\{1,3\}\}$ is not a subset of A; (g) $\{1,\alpha\}$ is an element of A, so $\{\{1,\alpha\}\}$ is a subset of A; (h) it is clear that $\{1,\alpha\}$ is an element of A; (i) the empty set $\emptyset$ is a subset of every set.
2. (a) Only the pair D and E have this property. To show that $D \not\subseteq E$, you need to give an example of an object that is an element of D, but not of E. An example would be $0 \in D$, but $0 \notin E$. To show that $E \not\subseteq D$, note that $1 \in E$, but $1 \notin D$.

   If you also want to show that this is really the only pair of sets with this property (among the sets given), note that $D \subseteq C \subseteq B$, and $E \subseteq C \subseteq B$.

   (b) Since $E \subseteq X$, we know that $X \neq D$ (from the previous problem), but it can still be any of the other sets.
3. (a) is not valid: the hypothesis for the first statement is not fulfilled, so we cannot conclude anything;

   (b) This statement is valid. Let P be the statement "I eat chocolate" and Q be the statement "I am depressed". Then the statements we are given become: $P \Rightarrow Q$ and $\overline{Q}$. From $P \Rightarrow Q$ we get $\overline{Q} \Rightarrow \overline{P}$. We have $\overline{Q}$, so we can conclude $\overline{P}$, that is, I am not eating chocolate.

   (c) This statement is valid. Let P be the statement "this movie is worth seeing", Q the statement "this movie is made in England" and R the statement "Ivor Smallbrain reviews this movie". Then the statements we were given translate to: $\overline{P} \Rightarrow \overline{Q}$, $P \Rightarrow R$, and $\overline{R}$. From $P \Rightarrow R$ we first get $\overline{R} \Rightarrow \overline{P}$, and with $\overline{R}$ we can conclude $\overline{P}$. The first statement now allows us to conclude $\overline{Q}$, so the movie was not made in England.

5 (a) In logical symbols, this statement reads "$n = 3 \Rightarrow n^2 - 2n - 3 = 0$". If we assume that $n = 3$, then $n^2 - 2n - 3 = 3^2 - 3 * 2 - 3 = 9 - 6 - 3 = 0$, so this statement is true.

(b) In logical symbols, this statement reads $n^2 - 2n - 3 = 0 \Rightarrow n = 3$. This is not true. For $n = -1$, the assumption is true: $1 + 2 - 3 = 0$, but the conclusion is not true.

(c) In logical symbols, this statement reads $n^2 - 2n - 3 = 0 \Rightarrow n = 3$. So this is not true for the same reason as in the previous part.

(d) In logical symbols, this statement reads

$$\forall a, b \in \mathbb{Z},\ (ab \text{ is a square }) \Rightarrow (a \text{ is a square} \wedge b \text{ is a square}),$$

or, even more formally,

$$\forall a, b \in \mathbb{Z},\ (\exists p, ab = p^2) \Rightarrow (\exists r, a = r^2 \wedge \exists s, b = s^2).$$

This statement is not true. To disprove it, we need to find two integers a and b such that ab is a square, but either a or b or both are not squares. And example would be $a = b = 3$. In this case $ab = 9$, which is a square, but neither a nor b is a square itself.

(e) In logical symbols, this statement reads

$$\forall a, b \in \mathbb{Z}, \ (a \text{ is a square} \wedge b \text{ is a square}) \Rightarrow (ab \text{ is a square }).$$

This statement is true. Suppose that a and b are squares, that is $a = p^2$ and $b = q^b$ for some integers p and q. Then $ab = p^2q^2 = (pq)^2$, which shows that ab is a square.

The answers to the truth table questions are:

(1) The truth table for $(P \Leftrightarrow Q) \Leftrightarrow [(P \wedge Q) \vee (\neg P \wedge \neg Q)]$ is as follows:

P	Q	$P \Leftrightarrow Q$	$P \wedge Q$	$\neg P$	$\neg Q$	$\neg P \wedge \neg Q$	$(P \wedge Q) \vee (\neg P \wedge \neg Q)$	$(P \Leftrightarrow Q) \Leftrightarrow [(P \wedge Q) \vee (\neg P \wedge \neg Q)]$
T	T	T	T	F	F	F	T	T
T	F	F	F	F	T	F	F	T
F	T	F	F	T	F	F	F	T
F	F	T	F	T	T	T	T	T

Since the last column only contains T's (so it is true regardless of the truth values of P and Q), we conclude that this statement is a tautology.

(2) Show that $P \Rightarrow (Q \vee R)$ is logically equivalent to $(P \Rightarrow Q) \vee (P \Rightarrow R)$. We create the truth table for both of these statements:

P	Q	R	$Q \vee R$	$P \Rightarrow (Q \vee R)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
F	T	T	T	T
T	F	F	F	F
F	T	F	T	T
F	F	T	T	T
F	F	F	F	T

P	Q	R	$P \Rightarrow Q$	$P \Rightarrow R$	$(P \Rightarrow Q) \vee (P \Rightarrow R)$
T	T	T	T	T	T
T	T	F	T	F	T
T	F	T	F	T	T
F	T	T	T	T	T
T	F	F	F	F	F
F	T	F	T	T	T
F	F	T	T	T	T
F	F	F	T	T	T

We see that the two last columns are exactly the same, so these two statements have the same truth value for each combination of truth values for P, Q, and R. We conclude that they are logically equivalent.

(3) To show that $(P \Rightarrow Q) \wedge (Q \Rightarrow \neg P)$ is not a contradiction, consider its truth table.

P	Q	$P \Rightarrow Q$	$\neg P$	$Q \Rightarrow \neg P$	$(P \Rightarrow Q) \wedge (Q \Rightarrow \neg P)$
T	T	T	F	F	F
T	F	F	F	T	F
F	T	T	T	T	T
F	F	T	T	T	T

We see here that this statement is true whenever P is false. in particular, it is not a contradiction.