Solutions to Assignment 4

7.1(a) To find the roots of $x^2 - 5x + 7 - i = 0$, we start by using the quadratic formula and we find $x = \frac{5 \pm \sqrt{-3+4i}}{2}$. Let $z = re^{i\theta}$ be one of those roots. Then $z^2 = -3 + 4i$. $|-3 + 4i| = \sqrt{9 + 16} = 5$ and the argument of $-3 + 4i$ is $\arccos(-3/5)$. This is not a nice angle, but that is not too important, we do know its sin, cos, and tan. Lets call $\varphi = \arccos(-3/5)$; then $\sin(\varphi) = 4/5$, $\cos(\varphi) = -3/5$ and $\tan(\varphi) = -4/3$. $z^2 = r^2 e^{2\theta i}$ so $r^2 = 5$ and $2\theta = \arccos(-3/5) = \varphi$, so $\theta = \frac{1}{2}\varphi$. So

$$\begin{aligned}
z &= \sqrt{5}(\cos(\theta) + i\sin(\theta)) \\
&= \sqrt{5}(\cos(\frac{1}{2}\varphi) + i\sin(\frac{1}{2}\varphi)) \\
&= \sqrt{5}(\sqrt{\frac{1+\cos(\varphi)}{2}} + i\sqrt{\frac{1-\cos(\varphi)}{2}}) \\
&= \sqrt{\frac{5}{2}}\left(\sqrt{1+\cos(\varphi)} + i\sqrt{1-\cos(\varphi)}\right) \\
&= \sqrt{\frac{5}{2}}\left(\sqrt{1+(-3/5)} + i\sqrt{1-(-3/5)}\right) \\
&= \sqrt{\frac{5}{2}}\left(\sqrt{2/5} + i\sqrt{8/5}\right) \\
&= 1 + 2i.
\end{aligned}$$

When we plug that back into our quadratic formula, we get $x = \frac{5 \pm (1+2i)}{2}$, so

$$x = 3 + i \text{ or } x = 2 - i.$$

We may check that these answers are correct: for $x = 3+i$, $x^2 - 5x + 7 - i = (3+i)^2 - 5(3+i) + 7 - i = 9 + 6i - 1 - 15 - 5i + 7 - i = 0$; and for $x = 2-i$, $x^2 - 5x + 7 - i = (2-i)^2 - 5(2-i) + 7 - i = 4 - 4i - 1 - 10 + 5i + 7 - i = 0$.

7.1(b) Note that this is really a quadratic equation in x^2, so we will first treat it as such and use the quadratic formula to get solutions for x^2.

$$x^2 = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{3}i}{2} = \frac{-1}{2} \pm \frac{\sqrt{3}}{2}i.$$

So we need to find x such that

$$x^2 = \frac{-1}{2} + \frac{\sqrt{3}}{2}i \text{ or } x^2 = \frac{-1}{2} - \frac{\sqrt{3}}{2}i.$$

We use the methods from before. First find a special root for $x^2 = \frac{-1}{2} + \frac{\sqrt{3}}{2}i$ suppose $x = re^{i\theta}$. Then $r = 1$ and $2\theta = \frac{2\pi}{3}$. So $\theta = \frac{2\pi}{6} = \frac{\pi}{3}$. So we get $x = e^{i\frac{\pi}{3}} = \cos(\pi/3) + i\sin(\pi/3) = \frac{1}{2} + i\frac{\sqrt{3}}{2}$. The other root is found by multiplying by a second root of unity, i.e., -1. So $x = \pm(\frac{1}{2} + i\frac{\sqrt{3}}{2})$. Now we treat $x^2 = \frac{-1}{2} - \frac{\sqrt{3}}{2}i$ in the same way. If $x = re^{i\theta}$, then $r = 1$ and $2\theta = -\frac{2\pi}{3}$, so $\theta = -\frac{\pi}{3}$. We find that $x = e^{-i\frac{\pi}{3}} = \frac{1}{2} - i\frac{\sqrt{3}}{2}$. The other root is then $x = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$. In summary, the roots are: $\frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} - i\frac{\sqrt{3}}{2}, \frac{1}{2} - i\frac{\sqrt{3}}{2}, -\frac{1}{2} + i\frac{\sqrt{3}}{2}$.

7.2 We use our algorithm for finding roots to find the roots of

$$x^3 - 6x^2 + 13x - 12 = 0.$$

The first step requires to make a substitution to get rid of the quadratic term. In this case $a = -6$, so we write $y = x + \frac{a}{3} = x - 6/3 = x - 2$, so we substitute $x = y + 2$. Then we get

$$(y+2)^3 - 6(y+2)^2 + 13(y+2) - 12 = 0.$$

Factoring this out using binomial coefficients gives

$$\binom{3}{0} y^3 + \binom{3}{1} y^2 * 2 + \binom{3}{2} y * 2^2 + \binom{3}{3} 2^3 - 6(y^2 + 4y + 4) + 13y + 26 - 12 = 0$$

and this can be rewritten as

$$y^3 + 6y^2 + 12y + 8 - 6y^2 - 24y - 24 + 13y + 14 = 0$$

and that can be rewritten as

$$y^3 + y - 2 = 0.$$

The next step of our algorithm is to find complex numbers u and v such that

$$-3uv = 1 \text{ and } -(u^3 + v^3) = -2$$

so

$$-3uv = 1 \text{ and } u^3 + v^3 = 2$$

Substituting $v = -\frac{1}{3u}$ in the second equation gives

$$u^3 - \frac{1}{27u^3} = 2$$

Multiplying by $27u^3$ gives

$$27u^6 - 1 + 2 * 27u^3 = 0$$

So

$$27u^6 + 54u^3 - 1 = 0$$

This is quadratic in u^3. Using the quadratic formula we find that one solution for u^3 is

$$u^3 = \frac{-54 + \sqrt{54^2 + 4 * 27}}{54} = \frac{-54 + \sqrt{3024}}{54} = \frac{-54 + 12\sqrt{21}}{54} = -1 + \frac{2}{9}\sqrt{21}$$

Now we need to solve for u. Note that u^3 is a real number, so we find one solution for u by simply taking the cube root of this number:

$$u = \sqrt[3]{\left(-1 + \frac{2}{9}\sqrt{21}\right)}$$

The other two solutions can be found by multiplying this solution by third roots of unity. Then we get

$$\begin{aligned} u &= \sqrt[3]{\left(-1 + \frac{2}{9}\sqrt{21}\right)} e^{i\frac{2\pi}{3}} \\ u &= \sqrt[3]{\left(-1 + \frac{2}{9}\sqrt{21}\right)} e^{i\frac{4\pi}{3}} \end{aligned}$$

We calculate the corresponding v-values and y-values:

$$u = \sqrt[3]{\left(-1 + \frac{2}{9}\sqrt{21}\right)} \text{ comes with } v = -\frac{1}{3}\left(-1 + \frac{2}{9}\sqrt{21}\right)^{-\frac{1}{3}}$$

and this gives

$$y = u + v = \sqrt[3]{\left(-1 + \frac{2}{9}\sqrt{21}\right)} - \frac{1}{3}\left(-1 + \frac{2}{9}\sqrt{21}\right)^{-\frac{1}{3}}$$

With a lot of clever algebra you can show that this gives $y = 1$, and the corresponding x-value is then: $x = 1 + 2 = 3$. (This can also be seen from the fact that you know that $x = 3$ must be a solution, and the other two solutions below are clearly complex numbers that are not real.) The other solutions are:

$$u = \sqrt[3]{\left(-1 + \frac{2}{9}\sqrt{21}\right)} e^{i\frac{2\pi}{3}} \text{ with } v = -\frac{1}{3}\left(-1 + \frac{2}{9}\sqrt{21}\right)^{-\frac{1}{3}} e^{-i\frac{2\pi}{3}}$$

This gives

$$\begin{aligned} y &= u + v \\ &= \sqrt[3]{\left(-1 + \frac{2}{9}\sqrt{21}\right)} e^{i\frac{2\pi}{3}} - \frac{1}{3}\left(-1 + \frac{2}{9}\sqrt{21}\right)^{-\frac{1}{3}} e^{-i\frac{2\pi}{3}} \end{aligned}$$

Note that the real part of this is

$$\begin{aligned} &\sqrt[3]{\left(-1 + \frac{2}{9}\sqrt{21}\right)} \cos(\frac{2\pi}{3}) - \frac{1}{3}\left(-1 + \frac{2}{9}\sqrt{21}\right)^{-\frac{1}{3}} \cos(-\frac{2\pi}{3}) = \\ &= -\frac{1}{2}\left(\sqrt[3]{\left(-1 + \frac{2}{9}\sqrt{21}\right)} - \frac{1}{3}\left(-1 + \frac{2}{9}\sqrt{21}\right)^{-\frac{1}{3}}\right) \\ &= -\frac{1}{2} \end{aligned}$$

So the real part of the corresponding value for x is $-\frac{1}{2} + 2 = \frac{3}{2}$. You can now plug in $\frac{3}{2} + bi$ as a root in the original equation and you find that $b = \pm\frac{1}{2}\sqrt{7}$, so we have the following solutions:

$$x = 3,\ x = \frac{3}{2} + \frac{1}{2}i\sqrt{7},\ x = \frac{3}{2} - \frac{1}{2}i\sqrt{7}$$

Note that the complex roots are complex conjugates. This is not coincidence! If the coefficients of a polynomial are real, the complex roots always come in pairs of complex conjugates.

8.2 *Claim* $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$ for all positive integers n.

Proof: We prove this by induction on n.

For the base case, let $n = 1$. Then the left hand side is equal to 1 and the right hand side is equal to $\frac{1}{6} * 1 * 2 * 3 = 1$, so this is correct.

For the induction hypothesis we assume that $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$

For the induction step we need to prove

$$\sum_{r=1}^{n+1} r^2 = \frac{1}{6}(n+1)(n+2)(2n+3)$$

We do this as follows:

$$\begin{aligned}
\sum_{r=1}^{n+1} r^2 &= \sum_{r=1}^{n} r^2 + (n+1)^2 \\
&= \frac{1}{6}n(n+1)(2n+1) + (n+1)^2 \quad \text{(by the induction hypothesis)} \\
&= \frac{1}{6}(n+1)(n(2n+1) + 6(n+1)) \\
&= \frac{1}{6}(n+1)(2n^2 + n + 6n + 6) \\
&= \frac{1}{6}(n+1)(2n^2 + 7n + 6) \\
&= \frac{1}{6}(n+1)(n+2)(2n+3)
\end{aligned}$$

as required. The result of our claim now follows by induction.

As a consequence of this result we prove a couple of related results:

$$\begin{aligned}
1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 &= \sum_{r=1}^{2n-1} r^2 - \sum_{r=1}^{n-1} (2r)^2 \\
&= \frac{1}{6}(2n-1)(2n)(4n-1) - \sum_{r=1}^{n-1} 4r^2 \\
&= \frac{1}{6}(2n-1)(2n)(4n-1) - 4\sum_{r=1}^{n-1} r^2 \\
&= \frac{1}{6}(2n-1)(2n)(4n-1) - \frac{1}{6}(n-1)(n)(2n-1) \\
&= \frac{1}{6}(2n-1)n(2(4n-1) + n - 1) \\
&= \frac{1}{6}(2n-1)n(8n - 2 + n - 1) \\
&= \frac{1}{6}(2n-1)n(9n-3) \\
&= \frac{3}{6}(2n-1)n(3n-1) \\
&= \frac{1}{2}n(2n-1)(3n-1)
\end{aligned}$$

$$\begin{aligned}
1\cdot 1+2\cdot 3+3\cdot 5+4\cdot 7+\cdots+n(2n-1) &= \sum_{r=1}^{n} r(2r-1)\\
&= \sum_{r=1}^{n}(2r^2-r)\\
&= \sum_{r=1}^{n} 2r^2-\sum_{r=1}^{n} r\\
&= 2\sum_{r=1}^{n} r^2-\frac{1}{2}n(n+1)\\
&= 2*\frac{1}{6}n(n+1)(2n+1)-\frac{1}{2}n(n+1)\\
&= \frac{2}{6}n(n+1)(2n+1)-\frac{3}{6}n(n+1)\\
&= \frac{1}{6}n(n+1)(2(2n+1)-3)\\
&= \frac{1}{6}n(n+1)(4n+2-3)\\
&= \frac{1}{6}n(n+1)(4n-1)
\end{aligned}$$

3(b) We guess that $\sum_{k=(n-1)^2+1}^{n^2} k=(n-1)^3+n^3$ for all positive integers n. There are several ways to prove this. One way goes as follows:

$$\begin{aligned}
\sum_{k=(n-1)^2+1}^{n^2} k &= \sum_{k=1}^{n^2} - \sum_{k=1}^{(n-1)^2} k\\
&= \frac{1}{2}n^2(n^2+1)-\frac{1}{2}(n-1)^2((n-1)^2+1)\\
&= \frac{1}{2}(n^4-(n-1)^4+n^2-(n-1)^2)\\
&= \frac{1}{2}((n^2-(n-1)^2)(n^2+(n-1)^2)+n^2-(n-1)^2)\\
&= \frac{1}{2}(n^2-(n-1)^2)(n^2+(n-1)^2+1)\\
&= \frac{1}{2}(2n-1)(2n^2-2n+2)\\
&= (2n-1)(n^2-n+1)\\
&= n^3+n^3-3n^2+3n-1\\
&= n^3+(n-1)^3
\end{aligned}$$

You can also consider this as follows:

$$\begin{aligned}
\sum_{k=(n-1)^2+1}^{n^2} k &= (n-1)^2+1+(n-1)^2+2+(n-1)^2+3+\cdots(n-1)^2+2n-1 \\
&= (2n-1)(n-1)^2+\sum_{k=1}^{2n-1} k \\
&= (2n-1)(n-1)^2+\frac{1}{2}(2n-1)(2n) \\
&= (2n-1)(n-1)^2+(2n-1)n \\
&= (2n-1)(n^2-2n+1+n) \\
&= (2n-1)(n^2-n+1) \\
&= n^3+(n-1)^3
\end{aligned}$$