ASYMPTOTICS OF A FAMILY OF BINOMIAL SUMS

ROB NOBLE

ABSTRACT. Using a recent method of Pemantle and Wilson, we study the asymptotics of a family of combinatorial sums that involve products of two binomial coefficients and includes both alternating and non-alternating sums. With the exception of finitely many cases the main terms are obtained explicitly, while the existence of a complete asymptotic expansion is established. A recent method by Flajolet and Sedgewick is used to establish the existence of a full asymptotic expansion for the remaining cases, and the main terms are again obtained explicitly. Among several specific examples we consider generalizations of the central Delannoy numbers and their alternating analogues.

1. INTRODUCTION

Some combinatorial sequences of interest can be written as binomial sums of the form

$$u_r^{(\varepsilon,a,d)} = \sum_{k=0}^{r} (-1)^{\varepsilon k} \binom{r}{k} \binom{ar}{k} d^k \tag{1}$$

for $\varepsilon \in \{0,1\}$ and $a, d \in \mathbb{N}$. For instance, the central binomial coefficients are given by

$$\binom{2r}{r} = u_r^{(0,1,1)} = \sum_{k=0}^{r} \binom{r}{k}^2$$

and the central Delannoy numbers $D(r,r)$ that count the number of paths from the origin $(0,0)$ to the point (r,r) using steps $(1,0)$, $(0,1)$ and $(1,1)$ are given by

$$D(r,r) = u_r^{(0,1,2)} = \sum_{k=0}^{r} \binom{r}{k}^2 2^k$$

(see, e.g., [4, p. 81], [13, p. 185]). Another sequence of interest, having $\varepsilon = 1$, is given by

$$u_r^{(1,2,1)} = \sum_{k=0}^{r} (-1)^k \binom{r}{k} \binom{2r}{k}.$$

The divisibility properties of this sequence are studied by Chamberland and Dilcher in [2] where it is shown that it behaves in many ways like a single binomial coefficient and, in particular, satisfies a version of Wolstenholme's Theorem. In [2], it is conjectured that this sequence possesses a full asymptotic expansion of a particular form as r tends to infinity. Here, we prove this conjecture and provide similar asymptotic expansions for the case of arbitrary ε, a and d in (1). Our approach will

---

Date: July 15, 2010.

Key words and phrases. Asymptotics, binomial sums, multivariate sequences, generalized Riordan arrays, central Delannoy numbers.

Research supported by the Natural Sciences and Engineering Research Council of Canada and Killam Trusts.

be to view the univariate sequence $\{u_r^{(\varepsilon,a,d)}\}_r$ as the diagonal of the bivariate sequence $\{\tilde{u}_{rs}^{(\varepsilon,a,d)}\}_{r,s}$ given by

$$\tilde{u}_{rs}^{(\varepsilon,a,d)} = \sum_{k=0}^{r} (-1)^{\varepsilon k} \binom{r}{k} \binom{as}{k} d^k. \tag{2}$$

There are two recent general methods for obtaining asymptotics for sequences of this type, namely the bivariate method of Pemantle and Wilson (see [12]) and the transfer method of Flajolet and Sedgewick (as developed in [5, Part B]). It will turn out that the method of Pemantle and Wilson can accommodate all but finitely many cases. We will then deal with the remaining cases by applying the transfer method of Flajolet and Sedgewick. For ease of notation, when the superscripts ε, a, d are understood, they will be omitted from the notation and we will write u_r and $\tilde{u}_{rs}$ instead of the more cumbersome $u_r^{(\varepsilon,a,d)}$ and $\tilde{u}_{rs}^{(\varepsilon,a,d)}$, respectively.

In order to state our main result, we require the concept of asymptotic series for sequences. An *asymptotic series* for a sequence $\{a_r\}_r$ is a formal series $\sum_\ell c_\ell r^{-\ell}$ such that, for all m, we have

$$a_r - \sum_{\ell < m} c_\ell r^{-\ell} = O(r^{-m}) \qquad (r \to \infty).$$

When an asymptotic series $\sum_\ell c_\ell r^{-\ell}$ exists for the sequence $\{a_r\}_r$, we write

$$a_r \sim \sum_\ell c_\ell r^{-\ell} \qquad (r \to \infty).$$

We also require the following notation. Let $a, d \in \mathbb{N}$ and $\varepsilon \in \{0, 1\}$. Set $\alpha = 1 - (-1)^\varepsilon d$, and define the polynomial g by

$$a(\alpha - 1)g(z) = \alpha z^2 + (a\alpha - a - \alpha - 1)z + 1. \tag{3}$$

Let

$$\Delta_g = \frac{(a-1)^2\alpha - (a+1)^2}{(\alpha - 1)a^2} \tag{4}$$

denote the discriminant of g, and z_0 be the root of g for which

$$\frac{2\alpha z_0 + a\alpha - a - \alpha - 1}{a(\alpha - 1)} = \sqrt{\Delta_g}$$

where $\sqrt{\cdot}$ denotes the principal branch of the square root. Further, define

$$\delta = \frac{1}{(1 - z_0)\sqrt[4]{\Delta_g}} \qquad \text{and} \qquad \beta = \frac{1}{z_0}\left(\frac{1 - \alpha z_0}{1 - z_0}\right)^a,$$

where $\sqrt[4]{\cdot}$ denotes the principal branch of the fourth root. The case when g has repeated real roots yields cube root asymptotics for u_r, while the other cases yield square root asymptotics for u_r. This gives rise to our main result which is split into two theorems to accommodate this distinction.

Theorem 1 ($\Delta_g \neq 0$ Case)**.** *With the above notation, there exist constants μ_ℓ for $\ell \in \mathbb{N}$ such that*

$$\sum_{k=0}^{r} (-1)^{\varepsilon k} \binom{r}{k} \binom{ar}{k} d^k \sim \frac{\delta \beta^r}{\sqrt{2\pi r}} \left(1 + \sum_{\ell=1}^{\infty} \frac{\mu_\ell}{r^\ell}\right) \qquad (r \to \infty),$$

in case $\Delta_g > 0$ *and*

$$\sum_{k=0}^{r}(-1)^{\varepsilon k}\binom{r}{k}\binom{ar}{k}d^k \sim \frac{\delta\beta^r}{\sqrt{2\pi r}}\left(1+\sum_{\ell=1}^{\infty}\frac{\mu_\ell}{r^\ell}\right)+\frac{\overline{\delta\beta}^r}{\sqrt{2\pi r}}\left(1+\sum_{\ell=1}^{\infty}\frac{\overline{\mu_\ell}}{r^\ell}\right) \qquad (r\to\infty),$$

in case $\Delta_g < 0$.

A calculation shows that $\Delta_g = 0$ only for $(\varepsilon, a, d) \in \{(1, 2, 8), (1, 3, 3)\}$, which accounts for the two cases in the following theorem.

Theorem 2 ($\Delta_g = 0$ Case)**.** *There exist constants* μ_ℓ, η_ℓ, $\tilde{\mu}_\ell$, $\tilde{\eta}_\ell \in \mathbb{Q}$ *for* $\ell \in \mathbb{N}$ *such that*

$$\sum_{k=0}^{r}(-1)^k\binom{r}{k}\binom{2r}{k}8^k \sim \frac{(-27)^r}{2^{2/3}\Gamma(2/3)r^{1/3}}\left(1+\sum_{\ell=1}^{\infty}\frac{\mu_\ell}{r^\ell}\right)+\frac{(-27)^r}{2^{4/3}\Gamma(1/3)r^{2/3}}\left(1+\sum_{\ell=1}^{\infty}\frac{\eta_\ell}{r^\ell}\right)\ (r\to\infty),$$

and

$$\sum_{k=0}^{r}(-1)^k\binom{r}{k}\binom{3r}{k}3^k \sim \frac{2^{2/3}(-16)^r}{3\Gamma(2/3)r^{1/3}}\left(1+\sum_{\ell=1}^{\infty}\frac{\tilde{\mu}_\ell}{r^\ell}\right)+\frac{2^{1/3}(-16)^r}{3\Gamma(1/3)r^{2/3}}\left(1+\sum_{\ell=1}^{\infty}\frac{\tilde{\eta}_\ell}{r^\ell}\right)\ (r\to\infty).$$

Asymptotics of binomial sums have been studied before. For instance, in [10], McIntosh established asymptotic expansions for sums of the form

$$\sum_{k=0}^{n}\binom{n}{k}^{r_0}\binom{n+k}{k}^{r_1}\binom{n+2k}{k}^{r_2}\cdots\binom{n+mk}{k}^{r_m}$$

as $n\to\infty$ for non-negative integers $r_0, r_1, r_2, \ldots, r_m$.

Our binomial sums of interest satisfy linear recurrence relations with polynomial coefficients. It was established by Birkhoff and Trjitzinsky in [1] that such sequences possess full asymptotic expansions of the sort considered here. However, this is not accepted as a theorem by experts. (See the remarks following [5, Theorem VIII.7], where the authors refer to discussions provided by Odlyzko [11, p. 1135–1138], Wimp [16, p. 64], and Wimp-Zeilberger [17] on this question). Our approach will be to study generating functions rather than the coefficient sequences directly, so that the well established asymptotic theory for differential and algebraic equations can be applied. Another reference that should be mentioned is [14], where, for several sequences of interest, the authors start with an asymptotic series and derive some divisibility properties for the coefficients of the series.

This paper is structured as follows. In Section 2 we give some preliminaries that set the stage for the remainder of the paper. After a few auxiliary results in Section 3, we deal, in Section 4, with the cases covered by the method of Pemantle and Wilson. In Section 5 we use the method of Flajolet and Sedgewick to consider the remaining finitely many cases. At that point, having established Theorem 1 and Theorem 2 in all cases, we conclude this paper with some examples in Section 6.

2. Preliminaries

The results in the sequel depend on the concept of asymptotic expansion. Since we will be dealing with more general expansions than asymptotic series for sequences, we open this section with the definitions required for this more general setting. We start with functions. Let $\zeta \in \mathbb{C}$, D be a domain containing ζ in its closure and denote by $\{G_k\}_k$ a sequence of functions for which, for all k, $G_{k+1}(z) = o(G_k(z))$ as $z \to \zeta,\ z \in D$. We say that the formal series $\sum_k G_k(z)$ is an *asymptotic expansion* of the function F as $z \to \zeta$, $z \in D$, and write

$$F(z) \sim \sum_k G_k(z) \qquad (z \to \zeta,\ z \in D)$$

provided, for all m,

$$F(z) - \sum_{k<m} G_k(z) = O(G_m(z)) \qquad (z \to \zeta,\ z \in D).$$

We define asymptotic expansions of sequences in an entirely analogous way. Suppose that $\{c_0(r)\}_r$, $\{c_1(r)\}_r$, $\{c_2(r)\}_r$, ... denote sequences in r for which, for all k, $c_{k+1}(r) = o(c_k(r))$ as $r \to \infty$. We say that the formal series $\sum_k c_k(r)$ is an *asymptotic expansion* of the sequence $\{a_r\}_r$ as $r \to \infty$, and write

$$a_r \sim \sum_k c_k(r) \qquad (r \to \infty)$$

provided, for all m,

$$a_r - \sum_{k<m} c_k(r) = O(c_m(r)) \qquad (r \to \infty).$$

Finally, consider bivariate sequences $\{a_{rs}\}_{r,s}$, $\{c_0(r,s)\}_{r,s}$, $\{c_1(r,s)\}_{r,s}$, $\{c_2(r,s)\}_{r,s}$, ... for which, for all k, $c_{k+1}(r,s) = o(c_k(r,s))$ as $r,s \to \infty$ (with r/s, s/r remaining bounded). We say that the formal series $\sum_k c_k(r,s)$ is an *asymptotic expansion* of the sequence $\{a_{rs}\}_{r,s}$ as $r,s \to \infty$ (with r/s, s/r remaining bounded), and write

$$a_{rs} \sim \sum_k c_k(r,s)$$

as $r,s \to \infty$ (with r/s, s/r remaining bounded), provided, for all m,

$$a_{rs} - \sum_{k<m} c_k(r,s) = O(c_m(r,s))$$

as $r,s \to \infty$ (with r/s, s/r remaining bounded).

We now proceed to the development of some preliminaries that are specific to our particular case of interest. Both the method of Flajolet and Sedgewick as well as the method of Pemantle and Wilson will proceed by analysis of the bivariate ordinary generating function

$$\tilde{F}(z,w) := \sum_{r,s\geq 0} \tilde{u}_{rs} z^r w^s.$$

Recall that we are setting $\alpha = 1 - (-1)^{\varepsilon} d$. If $\alpha = 0$, so that $\varepsilon = 0$ and $d = 1$, our sum is given by

$$\tilde{u}_{rs} = \sum_{k=0}^{r} \binom{r}{k}\binom{as}{k} = \binom{as+r}{r},$$

as a result of the Vandermonde convolution (see, e.g., [4, p. 44]). Since this case can be dealt with by way of Stirling's formula, we may suppose that $\alpha \neq 0$. Furthermore, as $d \neq 0$, we also have $\alpha \neq 1$. Our generating function is rational, as is shown by the following lemma.

Lemma 1. *Let $\varepsilon \in \{0,1\}$, $a, d \in \mathbb{N}$ and define $\alpha = 1 - (-1)^{\varepsilon} d$. With*

$$\tilde{u}_{rs} = \sum_{k=0}^{r} (-1)^{\varepsilon k} \binom{r}{k}\binom{as}{k} d^k \qquad \text{and} \qquad \tilde{F}(z,w) = \sum_{r,s\geq 0} \tilde{u}_{rs} z^r w^s,$$

we have

$$\tilde{F}(z,w) = \frac{\varphi(z)}{1 - w\nu(z)}$$

for

$$\varphi(z) = \frac{1}{1-z}, \qquad \nu(z) = \left(\frac{1-\alpha z}{1-z}\right)^a.$$

Proof. In order to compute the bivariate generating function $\tilde{F}(z,w) = \sum_{r,s\geq 0} \tilde{u}_{rs} z^r w^s$ of $\{\tilde{u}_{rs}\}_{r,s}$, observe that for sequences $\{a_r\}_r$ and $\{b_r\}_r$ such that

$$b_r = \sum_{k=0}^{r} \binom{r}{k} a_k \qquad (r \geq 0),$$

the ordinary generating functions $P(z) = \sum_{r=0}^{\infty} a_r z^r$ of $\{a_r\}_r$ and $Q(z) = \sum_{r=0}^{\infty} b_r z^r$ of $\{b_r\}_r$ are related by

$$Q(z) = \frac{1}{1-z} P\left(\frac{z}{1-z}\right).$$

This is related to Knuth's concept of (inverse) binomial transform (see [8]) as well as Flajolet's concept of binomial convolution (see [5, §II.2]). In our case, we find that

$$\sum_{r=0}^{\infty} \tilde{u}_{rs} z^r = \frac{1}{1-z} P\left(\frac{z}{1-z}\right),$$

where P is the ordinary generating function of

$$\left\{ \binom{as}{r} (1-\alpha)^r \right\}_r.$$

Since P is given by

$$P(z) = \sum_{r=0}^{\infty} \binom{as}{r} \left((1-\alpha)z\right)^r = (1 + (1-\alpha)z)^{as},$$

we find that

$$\sum_{r=0}^{\infty} \tilde{u}_{rs} z^r = \frac{1}{1-z} P\left(\frac{z}{1-z}\right) = \frac{1}{1-z}\left(1 + (1-\alpha)\frac{z}{1-z}\right)^{as} = \frac{1}{1-z}\left(\frac{1-\alpha z}{1-z}\right)^{as}.$$

Summing over s against w^s yields

$$\begin{aligned} \tilde{F}(z,w) = \sum_{r,s\geq 0} \tilde{u}_{rs} z^r w^s &= \frac{1}{1-z} \sum_{s \geq 0} \left[\left(\frac{1-\alpha z}{1-z}\right)^a w \right]^s \\ &= \frac{\frac{1}{1-z}}{1 - \left(\frac{1-\alpha z}{1-z}\right)^a w} = \frac{\varphi(z)}{1 - w\nu(z)}, \end{aligned} \tag{5}$$

where

$$\varphi(z) = \frac{1}{1-z}, \qquad \nu(z) = \left(\frac{1-\alpha z}{1-z}\right)^a.$$

This is as claimed. □

Since $\{u_r\}_r$ is the diagonal of a bivariate sequence having rational generating function, $F(x) = \sum_{r=0}^{\infty} u_r x^r$ is algebraic. This was first proved by Furstenberg in ([6]). In order to compute F, we will use the method given by Stanley ([13, p. 179]).

We rewrite $\tilde{F}(z,w)$ as

$$\tilde{F}(z,w) = \sum_{r,s\geq 0} \tilde{u}_{rs} z^r w^s = \frac{(1-z)^{a-1}}{(1-z)^a - w\,(1-\alpha z)^a};$$

then we substitute $w = x/z$ and divide by z to obtain

$$\frac{1}{z}\tilde{F}(z,x/z) = \sum_{r,s\geq 0} \tilde{u}_{rs} z^{r-s-1} x^s = \frac{(1-z)^{a-1}}{z(1-z)^a - x\,(1-\alpha z)^a}.$$

We see from here that

$$F(x) = \sum_{s=0}^{\infty} \tilde{u}_{ss} x^s$$

is the coefficient of z^{-1} in $\frac{1}{z}\tilde{F}(z,x/z)$. This is equal to the residue of $\frac{1}{z}\tilde{F}(z,x/z)$ at its unique pole $z(x)$ that tends to zero as x tends to zero. It is a simple pole and so the residue is obtained by evaluating the numerator at $z = z(x)$ and dividing by the derivative of the denominator evaluated at $z = z(x)$. This gives

$$F(x) = \frac{(1-z(x))^{a-1}}{(1-z(x))^{a-1}(1-(a+1)z(x)) + a\alpha x(1-\alpha z(x))^{a-1}}. \tag{6}$$

Once we find a polynomial $P(x,y)$ such that $P(x,F(x)) = 0$, we can use P to expand F into a Puiseux series about any chosen value of x. In particular, if we expand about the singularities of F having least nonzero modulus (the dominant singularities of F) then we can transfer the data appearing in these expansions by way of the singularity analysis of Flajolet and Sedgewick to obtain a full asymptotic expansion for u_r, valid as $r \to \infty$. The relevant definitions and results now follow.

Let ϕ and R be real numbers with $R > 1$ and $0 < \phi < \pi/2$. The open domain $\Delta(\phi,R)$ is defined as

$$\Delta(\phi,R) = \{z \in \mathbb{C} \mid |z| < R,\ z \neq 1,\ |\mathrm{Arg}(z-1)| > \phi\}.$$

A domain is a Δ-*domain at* 1 if it is equal to some $\Delta(\phi,R)$. For general nonzero $\zeta \in \mathbb{C}$, a Δ-*domain at* ζ is defined to be a set of the form $\zeta\Delta_0$ where Δ_0 is a Δ-domain at 1. The following result follows from the theory developed in Chapter VI of [5].

Proposition 1 (Flajolet, Sedgewick)**.** *Suppose that $\zeta_1,\ldots,\zeta_n$ are the dominant singularities of the ordinary generating function F of the sequence $\{a_r\}_r$. Suppose that F is analytic at the origin and that Δ_0 is a Δ-domain at* 1 *such that F is analytic in the domain*

$$D = \bigcap_{j=1}^{n} (\zeta_j \Delta_0).$$

If, for each j, F admits an expansion of the form

$$F(z) \sim \sum_{k\geq k_j} c_{j,k}(\zeta_j - z)^{\gamma_k} \qquad (z \to \zeta_j,\ z \in D),$$

then

$$a_r \sim \sum_{j=1}^{n} \sum_{k\geq k_j} c_{j,k}\zeta_j^{\gamma_k - r}\binom{r-\gamma_k-1}{r} \qquad (r \to \infty).$$

Proposition 1 always applies to algebraic generating functions and, in this case, the exponents γ_k are of the form k/κ for suitable $\kappa \in \mathbb{N}$ (see, e.g., [5, Theorem VII.7]). In our case, we will show that $F(x)$ admits an asymptotic expansion near each of its dominant singularities ζ that involves sums of the form

$$a_0(\zeta - x)^{-p/q}\left(1 + \sum_{k=1}^{\infty} c_k(\zeta - x)^k\right)$$

for suitable $p, q \in \mathbb{N}$. Defining $c_0 = 1$, this gives rise, by way of Proposition 1, to the asymptotic term

$$a_0\zeta^{-p/q-r}\sum_{k=0}^{\infty} c_k\zeta^k\binom{-k+p/q+r-1}{r}.$$

Now,

$$\binom{-k+p/q+r-1}{r} \sim \frac{r^{p/q-k-1}}{\Gamma(p/q-k)}\sum_{j=0}^{\infty}\frac{e_j(p/q-k)}{r^j} \qquad (r \to \infty)$$

where $e_0(x) = 1$ and $e_j(x) \in \mathbb{Q}[x]$ is of degree $2j$ and divisible by $x(x-1)\ldots(x-j)$. In fact, we have

$$e_j(x) = \sum_{\ell=j}^{2j}\lambda_{k\ell}(x-1)(x-2)\ldots(x-\ell),$$

where $\lambda_{k\ell} \in \mathbb{Q}$ is the coefficient of $v^k t^\ell$ in the power series expansion of $e^t(1+vt)^{-1-1/v}$ (see [5, Note VI.3, p. 384]). Our asymptotic term can therefore be written as

$$a_0\zeta^{-p/q-r}r^{p/q-1}\sum_{k,j\geq 0}\frac{c_k\zeta^k e_j(p/q-k)}{\Gamma(p/q-k)r^{j+k}} = a_0\zeta^{-p/q-r}r^{p/q-1}\sum_{\ell=0}^{\infty}\frac{h_\ell}{r^\ell},$$

where

$$h_\ell = \sum_{k=0}^{\ell}\frac{c_k\zeta^k e_{\ell-k}(p/q-k)}{\Gamma(p/q-k)} \qquad (\ell \geq 0).$$

Factoring out the leading term yields

$$\frac{a_0\zeta^{-p/q-r}r^{p/q-1}}{\Gamma(p/q)}\left(1 + \sum_{\ell=1}^{\infty}\frac{\Gamma(p/q)h_\ell}{r^\ell}\right).$$

Finally, we can apply the functional equation $\Gamma(z) = (z-1)\Gamma(z-1)$ repeatedly to find that, for $m \in \mathbb{N}$,

$$\frac{\Gamma(z)}{\Gamma(z-m)} = (z-1)_m$$

where

$$(\cdot)_k := \binom{\cdot}{k}k!$$

denotes the falling Pochhammer symbol. This shows that

$$\begin{aligned}\Gamma(p/q)h_\ell &= \sum_{k=0}^{\ell}\frac{\Gamma(p/q)c_k\zeta^k e_{\ell-k}(p/q-k)}{\Gamma(p/q-k)} \\ &= \sum_{k=0}^{\ell}c_k(p/q-1)_k\zeta^k e_{\ell-k}(p/q-k) \in \mathbb{Q}(\zeta, c_1, \ldots, c_\ell).\end{aligned}$$

This allows us to rewrite the asymptotic term in question as

$$\frac{a_0\zeta^{-p/q-r}r^{p/q-1}}{\Gamma(p/q)}\left(1+\sum_{\ell=1}^{\infty}\frac{\mu_\ell}{r^\ell}\right), \qquad (\mu_\ell \in \mathbb{Q}(\zeta, c_1, \ldots, c_\ell)). \tag{7}$$

Now, we will show that $y = F(x)$ satisfies a polynomial $P(x, y) \in \mathbb{Q}[x, y]$ of degree $a + 1$ in y. It will follow that $F(x)$ satisfies a linear ordinary differential operator with coefficients in $\mathbb{Q}[x]$ of order $a + 1$ (see, e.g., [13, Theorem 6.4.6]). By the method of Frobenius (see, e.g., [3, §4.8]), the expression

$$(\zeta - x)^{-p/q}\left(1+\sum_{k=1}^{\infty}c_k(\zeta - x)^k\right)$$

will be a series solution to the corresponding ordinary differential equation which will lead to a linear recurrence relation for the c_k of the form

$$\sum_{j=0}^{m} Q_j(k)c_{k+j} = 0 \qquad (k \geq 0),$$

for some m and suitable polynomials $Q_0(x), Q_1(x), \ldots, Q_m(x) \in \mathbb{Q}(\zeta)[x]$ with $Q_m \neq 0$. From this we conclude that all of the c_k lie in $\mathbb{Q}(\zeta)$ provided that $c_1, c_2, \ldots, c_{m-1}$ lie in $\mathbb{Q}(\zeta)$. For each of the finitely many cases that remain after applying the methods of Pemantle and Wilson, we show that this is indeed the case and conclude that $c_k \in \mathbb{Q}(\zeta)$ for all k. Since ζ will lie in $\mathbb{Q}(\sqrt{\Delta_g})$, we conclude ultimately that $c_k \in \mathbb{Q}(\sqrt{\Delta_g})$ for all k. From (7), we then have $\mu_\ell \in \mathbb{Q}(\sqrt{\Delta_g})$ for all ℓ as well.

3. Some auxiliary results on F

3.1. Computation of the dominant singularities of F. Our simple pole $z = z(x)$ satisfies

$$x = \frac{z(1-z)^a}{(1-\alpha z)^a}.$$

Also, from (6) we see that for this value of z, we have

$$F(x) = \frac{(1-z(x))^{a-1}}{(1-z(x))^{a-1}(1-(a+1)z(x)) + a\alpha x(1-\alpha z(x))^{a-1}}.$$

If we eliminate x, with $y = F(x)$, we have the parametric equations

$$x = \frac{z(1-z)^a}{(1-\alpha z)^a}, \qquad y = \frac{1-\alpha z}{p(z)}, \qquad p(z) = \alpha z^2 + (a\alpha - a - \alpha - 1)z + 1. \tag{8}$$

We can therefore determine the singularities of F by computing $\frac{dy}{dx}$ implicitly. We obtain

$$F'(x) = \frac{-(1-\alpha z)^{a+1}q(z)}{(1-z)^a p(z)^3}$$

where

$$q(z) = \alpha^2 z^3 - \alpha(\alpha+2)z^2 + (a + 1 - a\alpha + 2\alpha)z + a\alpha - a - 1.$$

Now, since we seek the singularities of least nonzero modulus and $x = 0$ when $z = 1$ and $x \to \infty$ as $z \to \frac{1}{\alpha}$, we can exclude these values of z from contention. Also, if p and q share a root then their resultant, given by

$$a^2\alpha^2(1-\alpha)^3((a-1)^2\alpha - (a+1)^2)$$

would have to vanish. Since $a \in \mathbb{N}$ and $\alpha \notin \{0,1\}$, this would force $(a-1)^2\alpha - (a+1)^2 = 0$, so that

$$\alpha = \left(\frac{a+1}{a-1}\right)^2.$$

But this forces p to have a double root at $z = \frac{1-a}{1+a}$ which then appears in the denominator with multiplicity 6. Since it appears as a root of q with multiplicity at most 3, it follows that, in any case, the roots of $p(z) = \alpha z^2 + (a\alpha - a - \alpha - 1)z + 1$ are singularities. In case this polynomial has complex conjugate roots, both roots correspond to dominant singularities while in case this polynomial has real roots, the corresponding value of x having smaller absolute value is the unique dominant singularity.

3.2. **The polynomial $P(x, y)$ satisfied by F.** In order to find $P(x, y)$, we eliminate z from the parametric equations given by (8). This is done by calculating the resultant of

$$p(z)y - (1 - \alpha z) \qquad \text{and} \qquad (1 - \alpha z)^a x - z(1 - z)^a \tag{9}$$

with respect to z, where

$$p(z) = \alpha z^2 + (a\alpha - a - \alpha - 1)z + 1 = (1 - z)(1 - \alpha z) - a(1 - \alpha)z.$$

This resultant is given by

$$R(x, y) = (\alpha y)^{a+1} \prod_{j=1}^{2} [(1 - \alpha z_j(y))^a x - z_j(y)(1 - z_j(y))^a]$$

where $z_1(y)$ and $z_2(y)$ are the roots of $p(z)y - (1 - \alpha z)$ (see, e.g., [7, Ch. 12]). A calculation using the computer algebra system Maple 11 (see [9]) determines that

$$R(x, y) = a^a \alpha^{a+1}(\alpha - 1)^a y^{a+1} x^2 + S(y)x + (\alpha - 1)^a (y - 1)(ay + 1)^a. \tag{10}$$

where

$$\begin{aligned} S(y) &= \frac{(\alpha-1)^a}{2^{a+1}} \left((L^-(y) - \sqrt{\Delta(y)})(L^+(y) + \sqrt{\Delta(y)})^a + (L^-(y) + \sqrt{\Delta(y)})(L^+(y) - \sqrt{\Delta(y)})^a \right) \\ &= \frac{(\alpha-1)^a}{2^a} \left(L^-(y) \sum_k \binom{a}{2k} L^+(y)^{a-2k} \Delta(y)^k - \Delta(y) \sum_k \binom{a}{2k+1} L^+(y)^{a-2k-1} \Delta(y)^k \right), \end{aligned}$$

$$L^+(y) = (\alpha(a - 1) + (a + 1))y + \alpha, \qquad L^-(y) = (\alpha(a - 1) - (a + 1))y + \alpha,$$

and

$$\begin{aligned} \Delta(y) &= (\alpha - 1)((a - 1)^2\alpha - (a + 1)^2)y^2 + 2\alpha(a - 1)(\alpha - 1)y + \alpha^2 \\ &= L^+(y)^2 - 4a\alpha y(ay + 1) = L^-(y)^2 - 4\alpha y(y - 1). \end{aligned}$$

We then have $P(x, F(x)) = 0$, where we set

$$P(x, y) = \frac{R(x, y)}{(\alpha - 1)^a} = a^a \alpha^{a+1} y^{a+1} x^2 + \frac{S(y)}{(\alpha - 1)^a} x + (y - 1)(ay + 1)^a.$$

In particular, the dominant singularities satisfy the resultant of $p(z)$ and $(1 - \alpha z)^a x - z(1 - z)^a$, which is the leading term of $R(x, y)$ as a polynomial in y. Using Maple to compute the coefficient of y in $y^{a+1} R(x, 1/y)$ we find that the coefficient of y^a in $R(x, y)$ equals 0. Also, we have

$$R(x, 0) = -(\alpha - 1)^a.$$

Therefore, with

$$\{\zeta_1, \zeta_2\} = \left\{ \frac{z(1-z)^a}{(1-\alpha z)^a} \;\middle|\; p(z) = 0 \right\},$$

we have

$$\begin{aligned} P(x,y) = \frac{R(x,y)}{(\alpha-1)^a} &= a^a \alpha^{a+1} y^{a+1} x^2 + \frac{S(y)}{(\alpha-1)^a} x + (y-1)(ay+1)^a \\ &= a^a \alpha^{a+1} (x-\zeta_1)(x-\zeta_2) y^{a+1} - \sum_{k=1}^{a-1} L_k^{(a,\alpha)}(x) y^k - 1 \end{aligned} \tag{11}$$

for suitable linear polynomials $L_k^{(a,\alpha)}(x) \in \mathbb{Q}[x]$. A further calculation shows that $L_{a-1}^{(a,\alpha)}(x) \neq 0$. Being unable to explicitly determine the $L_k^{(a,\alpha)}(x)$ for general $a \in \mathbb{N}$, we turn to the method of Pemantle and Wilson in order to eliminate all but finitely many cases. We will then compute $P(x,y)$ explicitly, on an individual basis, for the finitely many cases that remain.

4. The cases covered by Pemantle and Wilson

4.1. **Preliminary results of Pemantle and Wilson.** Bivariate sequences $\{a_{rs}\}_{r,s}$ having generating function $\tilde{F}(z,w)$ of the form

$$\tilde{F}(z,w) = \sum_{r,s \geq 0} a_{rs} z^r w^s = \frac{\varphi(z)}{1 - w\nu(z)}$$

for meromorphic functions φ and ν that are analytic at $z = 0$ are called *generalized Riordan arrays* (see, e.g., [15]). We see from (5) that the binomial sums we are considering are of this type. Using the multivariate methods developed by Pemantle and Wilson in [12], we can obtain a full asymptotic expansion for such sequences, valid in suitable directions determined by the simple poles of $\tilde{F}$ that are minimal in a sense to be described below. In [15], Wilson determined the leading terms of an expansion in case there exists one, and showed that if the sequence consists entirely of non-negative numbers, then there is a unique simple pole determining a direction in which we obtain an asymptotic expansion. Before stating the relevant results, we need to define the set S_{rs} of points that determine the directions of expansion. First of all, we say that a pole (z_0, w_0) of $\tilde{F}$ (so that $w_0 = \nu(z_0)^{-1}$) is *minimal* if every pole that lies in the closed bi-disk determined by (z_0, w_0) in fact lies in the torus determined by (z_0, w_0). That is, a pole (z_0, w_0) of $\tilde{F}$ is minimal provided that for all poles (z, w) of $\tilde{F}$, we have

$$|z| \leq |z_0| \text{ and } |w| \leq |w_0| \implies |z| = |z_0| \text{ and } |w| = |w_0|.$$

The set S_{rs} is then given by

$$\begin{aligned} S_{rs} = \{z \in \mathbb{C} \mid\ &(z, \nu(z)^{-1}) \text{ is minimal, } \varphi(z) \neq 0, \\ &sz\nu'(z) = r\nu(z) \text{ and } sz\nu''(z) \neq (r-s)\nu'(z)\}. \end{aligned} \tag{12}$$

The condition $sz\nu'(z) = r\nu(z)$ comes from the requirement that

$$[r, s] = [zH_z(z,w), wH_w(z,w)] \in \mathbb{P}^1,$$

where $H(z,w) = 1 - w\nu(z)$. This is the direction along which we obtain our asymptotic expansion for $r, s \to \infty$. In our example of interest, $\nu(0) \neq 0$ and since $\alpha \neq 1$, $\nu(z)$ is not a polynomial. In order to simplify the statements of the relevant results from [12] and [15], we will add these hypotheses. The first result combines Theorems 3.1, 3.3, and Corollary 3.7 of [12] to obtain the existence of the expansion with [15] to determine the leading terms.

Proposition 2. *Let $\{a_{rs}\}_{r,s}$ denote a bivariate sequence of complex numbers with ordinary generating function $\tilde{F}$ given by*

$$\tilde{F}(z,w) = \sum_{r,s\geq 0} a_{rs} z^r w^s = \frac{\varphi(z)}{1 - w\nu(z)},$$

for some meromorphic functions φ and ν that are analytic at $z = 0$. Suppose further that ν is not a polynomial and $\nu(0) \neq 0$. Let S_{rs} be defined by (12) and suppose that S_{rs} is finite and nonempty. Then there exist constants $c_\ell^{(z_{rs})}$ for $\ell \in \mathbb{N}$ and $z_{rs} \in S_{rs}$ such that

$$a_{rs} \sim \sum_{z_{rs}\in S_{rs}} \frac{\varphi(z_{rs})\nu(z_{rs})^s}{z_{rs}^r \sqrt{2\pi s Q_{rs}(z_{rs})}} \left(1 + \sum_{\ell=1}^{\infty} \frac{c_\ell^{(z_{rs})}}{s^\ell}\right)$$

as $r, s \to \infty$ (with r/s, s/r remaining bounded), where $\sqrt{\cdot}$ denotes the principal branch of the square root and

$$Q_{rs}(z) = \frac{z^2\nu''(z)}{\nu(z)} - \frac{r(r-s)}{s^2}.$$

In [15], the author shows that in case $a_{rs} \geq 0$ for all r and s, S_{rs} is a singleton, consisting of a single positive real number less than the radius of convergence ρ of ν. In this case, we obtain the following corollary of Proposition 2.

Corollary 1. *With notation as in Proposition 2, let $\rho > 0$ denote the radius of convergence of ν and suppose further that ν is not a polynomial and $\nu(0) \neq 0$. Let S_{rs} be defined by (12). Then $S_{rs} = \{x_{rs}\}$ for some $0 < x_{rs} < \rho$ and there exist constants $c_\ell^{(r,s)}$ for $\ell \in \mathbb{N}$ such that*

$$a_{rs} \sim \frac{\varphi(x_{rs})\nu(x_{rs})^s}{x_{rs}^r \sqrt{2\pi s Q_{rs}(x_{rs})}} \left(1 + \sum_{\ell=1}^{\infty} \frac{c_\ell^{(r,s)}}{s^\ell}\right)$$

as $r, s \to \infty$ (with r/s, s/r remaining bounded), where $\sqrt{\cdot}$ denotes the principal branch of the square root and

$$Q_{rs}(z) = \frac{z^2\nu''(z)}{\nu(z)} - \frac{r(r-s)}{s^2}.$$

Remark 1. A more general result due to Pemantle and Wilson allows for the case where $Q_{rs}(x_{rs}) = 0$ (see [12, Theorem 3.3]). However, in this case, we only obtain smooth minimal points in a valid direction for expansion in case $s = r$ which corresponds to Theorem 2. Although we will prove Theorem 2 by the method of Flajolet and Sedgewick, it should be noted that the leading terms obtained in Theorem 2 agree with what is predicted by this more general result of Pemantle and Wilson.

4.2. **Applying the methods of Pemantle and Wilson.** We are interested in the asymptotics of the binomial sums

$$\tilde{u}_{rs} = \sum_{k=0}^{r} \binom{r}{k}\binom{as}{k}(1-\alpha)^k,$$

as r and s tend to infinity in a suitable direction. By setting $s = ar$ in the bivariate asymptotic expansions obtained, we may suppose that $a = 1$. We then have

$$\varphi(z) = \frac{1}{1-z}, \qquad \nu(z) = \frac{1-\alpha z}{1-z}.$$

Since $\varphi(z) \neq 0$ for any z satisfying $1 - w\nu(z) = 0$, the set S_{rs} defined by (12) is given by

$$S_{rs} = \{z \in \mathbb{C} \mid (z, \nu(z)^{-1}) \text{ is minimal, } sz\nu'(z) = r\nu(z),\ sz\nu''(z) \neq (r-s)\nu'(z)\}.$$

But

$$\frac{z\nu'(z)}{\nu(z)} = \frac{1}{1-z} - \frac{1}{1-\alpha z}, \qquad \frac{z\nu''(z)}{\nu'(z)} = \frac{2z}{1-z}.$$

Denoting the set of minimal points by $\mathcal{M}$, we can therefore rewrite the conditions of membership in the set S_{rs} as $(z, \nu(z)^{-1}) \in \mathcal{M}$ and

$$\mu(z) := \frac{1}{1-z} - \frac{1}{1-\alpha z} = \frac{r}{s}, \qquad \frac{2z}{1-z} \neq \frac{r}{s} - 1.$$

The second condition is equivalent to $z \neq (r-s)/(r+s)$, but this follows from the first equation since if $z = (r-s)/(r+s)$, the first equation forces $\alpha = (r+s)^2/(r-s)^2$ which fails to be a constant. Defining f_{rs} by

$$r\alpha f_{rs}(z) = (1-z)(1-\alpha z)(r - s\mu(z)) = r\alpha z^2 - ((1+\alpha)r + (1-\alpha)s)z + r,$$

we can rewrite S_{rs} as

$$S_{rs} = \{z \in \mathbb{C} \mid (z, \nu(z)^{-1}) \in \mathcal{M} \text{ and } f_{rs}(z) = 0\}. \tag{13}$$

From now on, we will denote the roots of f_{rs} by z_{rs}^+ and z_{rs}^-, where we have labelled the roots so that $z_{rs}^+ - z_{rs}^- = \sqrt{\Delta_{f_{rs}}}$ where $\Delta_{f_{rs}}$ denotes the discriminant of f_{rs} and $\sqrt{\cdot}$ denotes the principal branch of the square root. Also, the main terms of the asymptotic expansions appearing in the statement of Proposition 2 are given by

$$\frac{\varphi(z_{rs}^\pm)\nu(z_{rs}^\pm)^s}{(z_{rs}^\pm)^r\sqrt{2\pi s Q_{rs}(z_{rs}^\pm)}}, \qquad \text{where} \qquad Q_{rs}(z) = \frac{z^2\nu''(z)}{\nu(z)} - \frac{r(r-s)}{s^2}.$$

A calculation shows that

$$\begin{aligned} sQ_{rs}(z_{rs}^\pm) &= s\left[\frac{(z_{rs}^\pm)^2\nu''(z_{rs}^\pm)}{\nu(z_{rs}^\pm)} + \frac{z_{rs}^\pm\nu'(z_{rs}^\pm)}{\nu(z_{rs}^\pm)} - \left(\frac{z_{rs}^\pm\nu'(z_{rs}^\pm)}{\nu(z_{rs}^\pm)}\right)^2\right] \\ &= \frac{s(1-\alpha)z_{rs}^\pm(1-\alpha(z_{rs}^\pm)^2)}{(1-z_{rs}^\pm)^2(1-\alpha z_{rs}^\pm)^2} = \frac{r^2}{s(1-\alpha)}\left[\frac{1-\alpha(z_{rs}^\pm)^2}{z_{rs}^\pm}\right]. \end{aligned}$$

But the product of the roots of f_{rs} is equal to $1/\alpha$ and so

$$z_{rs}^\pm(z_{rs}^\pm \mp \sqrt{\Delta_{f_{rs}}}) = \frac{1}{\alpha} \qquad \text{or} \qquad \frac{1-\alpha(z_{rs}^\pm)^2}{z_{rs}^\pm} = \mp\alpha\sqrt{\Delta_{f_{rs}}}.$$

Therefore, we have

$$sQ_{rs}(z_{rs}^\pm) = \frac{r^2}{s(1-\alpha)}\left[\frac{1-\alpha(z_{rs}^\pm)^2}{z_{rs}^\pm}\right] = \pm\frac{r^2\alpha\sqrt{\Delta_{f_{rs}}}}{s(\alpha-1)}.$$

The leading terms of the expansion then become

$$\frac{\varphi(z_{rs}^\pm)\nu(z_{rs}^\pm)^s}{(z_{rs}^\pm)^r\sqrt{\pm 2\pi\frac{r^2\alpha\sqrt{\Delta_{f_{rs}}}}{s(\alpha-1)}}} = \frac{(1-\alpha z_{rs}^\pm)^s}{r(z_{rs}^\pm)^r(1-z_{rs}^\pm)^{s+1}}\sqrt{\pm\frac{(\alpha-1)s}{2\pi\alpha\sqrt{\Delta_{f_{rs}}}}}. \tag{14}$$

Finally, we need to determine the set S_{rs}. If $\varepsilon = 0$ so that $\alpha < 0$, then $\tilde{u}_{rs} \geq 0$ for all r and s and so Corollary 1 applies and we can conclude that S_{rs} is a singleton, consisting of a single positive real number less than one. By graphing the curve

$$\mu(x) = \frac{1}{1-x} - \frac{1}{1-\alpha x},$$

it is seen that for any $r, s > 0$, $\mu(x) = r/s$ has two solutions, one lying between 0 and 1 and the other being negative and less than $1/\alpha$. It follows that $S_{rs} = \{x_{rs}\}$ where $x_{rs} = z^+_{rs}$. Replacing α with $1 - d$ yields the following result.

Proposition 3. *Let $d \in \mathbb{N}$. The polynomials f_{rs} given by*

$$r(1-d)f_{rs}(z) = (1-d)rz^2 + ((d-2)r - ds)z + r,$$

have distinct real roots $x^+_{rs} > x^-_{rs}$. Define $x_{rs} = x^+_{rs}$. Then $0 < x_{rs} < 1$ and there exist constants $c^{(r,s)}_\ell$ for $\ell \in \mathbb{N}$ such that

$$\sum_{k=0}^{r} \binom{r}{k}\binom{s}{k} d^k \sim \frac{(1-(1-d)x_{rs})^s}{r x_{rs}^r (1-x_{rs})^{s+1}} \sqrt{\frac{ds}{2\pi(d-1)\sqrt{\Delta_{f_{rs}}}}} \left(1 + \sum_{\ell=1}^{\infty} \frac{c^{(r,s)}_\ell}{s^\ell}\right)$$

as $r, s \to \infty$ (with r/s, s/r remaining bounded), where $\sqrt{\cdot}$ denotes the principal branch of the square root.

We now turn to the alternating case given by $\varepsilon = 1$. This corresponds to the case $\alpha > 1$. We need to determine whether 0, 1 or 2 of the roots of f_{rs} give rise to minimal points. Define

$$\gamma(z) = \frac{1}{\nu(z)} = \frac{1-z}{1-\alpha z}.$$

Every point of $\mathcal{M}$ has first coordinate z such that $\gamma(z)$ realizes the minimum modulus of the points in the image of the closed disk determined by z under γ. That is, if $(z, w(z))$ is minimal and we define D_t for $t > 0$ to be the image of the closed disk of radius t centred at the origin, we have

$$|\gamma(z)| = \min\{|w| : w \in D_{|z|}\}.$$

We now turn to the determination of such points. We will use the fact that γ is a Möbius transformation defined on the extended complex plane $\mathbb{P}^1(\mathbb{C})$ and as such sends disks to disks, preserving their boundary circles. Let $t > 0$, and consider the circle centred at the origin with radius t. Since

$$\gamma(t) = \frac{1-t}{1-\alpha t}, \tag{15}$$

$$\gamma(ti) = \frac{(1+\alpha t^2) + i(\alpha-1)t}{1+\alpha^2 t^2}, \tag{16}$$

$$\gamma(-t) = \frac{1+t}{1+\alpha t}, \tag{17}$$

we see that the image of the circle in question is the unique circle in $\mathbb{P}^1(\mathbb{C})$ passing through the points (15), (16) and (17). This is easily seen to be the unique circle C_t in $\mathbb{P}^1(\mathbb{C})$ having centre lying on the extended real axis $\mathbb{P}^1(\mathbb{R})$ for which $C_t \cap \mathbb{P}^1(\mathbb{R}) = \{\gamma(-t), \gamma(t)\}$. In case $t = 1/\alpha$, this circle is given by $C_{1/\alpha} = \{z \in \mathbb{C} \mid \Re(z) = \frac{\alpha+1}{2\alpha}\} \cup \{\infty\} \subseteq \mathbb{P}^1(\mathbb{C})$. Now, each circle in $\mathbb{P}^1(\mathbb{C})$ is the boundary circle of two disks in $\mathbb{P}^1(\mathbb{C})$. Indeed, the exterior of any disk is itself a disk having the same boundary circle. The image of the open disk centred at the origin with radius t will be the open disk in $\mathbb{P}^1(\mathbb{C})$ with boundary circle C_t that contains $\gamma(0) = 1$. Its closure will be the

Range for t	Ordering of 0, 1, $\gamma(t)$, $\gamma(-t)$
$0 < t < \frac{1}{\alpha}$	$0 < \gamma(-t) < 1 < \gamma(t)$
$\frac{1}{\alpha} < t < 1$	$\gamma(t) < 0 < \gamma(-t) < 1$
$t > 1$	$0 < \gamma(t) < \gamma(-t) < 1$

TABLE 1. Ordering of γ values.

previously defined closed disk D_t. Suppose that $(z, w(z))$ is minimal. Since $1 - w\nu(z) = 0$ we see that $z \neq 1$ so that $\gamma(z) \neq 0$. Letting $|z| = t$, we see that

$$0 \neq |\gamma(z)| = \min\{|w| : w \in D_t\},$$

so that $0 \notin D_t$. Since $1 \in D_t$, we conclude that in order to obtain a minimal point having first coordinate z with modulus t, we require exactly one of 0, 1 to lie between $\gamma(-t)$ and $\gamma(t)$. Also, when this is the case, $z = \pm t$ unless C_t is centred at the origin and has radius less than 1. Indeed, since C_t is centred on the real axis, we see that the minimum modulus of points on C_t occurs at one of $\gamma(t)$, $\gamma(-t)$ and only occurs at additional points if C_t is centred at the origin. This latter case occurs when $\gamma(t) = -\gamma(-t)$ which a calculation shows to occur when $t = 1/\sqrt{\alpha}$. Since $Q_{rs}(z) \neq 0$, we are excluding $\pm\frac{1}{\sqrt{\alpha}}$, and so we obtain in this case that $|z| = 1/\sqrt{\alpha}$, $z \in \mathbb{C} \setminus \mathbb{R}$. A calculation provides us with the information found in Table 1. An inspection of Table 1 shows that we fail to obtain minimal points when $t > 1$ and obtain minimal points otherwise. Finally, we need to determine, for $t < 1$, which of $\gamma(t)$, $\gamma(-t)$ is closer to the origin. If $\gamma(-t) \neq -\gamma(t)$ then we obtain a unique minimal point. We obtain the possible minimal points described in Table 2. Also, in the limiting case $t \to \frac{1}{\alpha}$, the image of $|z| = t$ under γ is equal to $\Re(z) = \frac{\alpha+1}{2\alpha}$. We therefore obtain minimal points for this modulus since $0 < \frac{\alpha+1}{2\alpha} < 1$ when $\alpha > 1$. The minimal point obtained in this case is given by $\left(-\frac{1}{\alpha}, \frac{\alpha+1}{2\alpha}\right)$. Putting this all together gives the following characterization of the set $\mathcal{M}$ of minimal points:

Proposition 4. *For $\alpha > 1$ and excluding $\pm 1/\sqrt{\alpha}$, the set of minimal points is given by*

$$\left\{(x, \gamma(x)) \;\middle|\; -\frac{1}{\sqrt{\alpha}} < x < 0 \text{ or } \frac{1}{\sqrt{\alpha}} < x < 1\right\} \cup \left\{(z, \gamma(z)) \;\middle|\; |z| = \frac{1}{\sqrt{\alpha}},\ z \in \mathbb{C} \setminus \mathbb{R}\right\}.$$

Proof. We showed above that these are the only possibilities for minimal points. What needs to be shown here is that each of these candidates is in fact minimal. In each case, we know that for our candidate $(z, w(z))$, we have

$$|\gamma(z)| = \min\{|\gamma(z')| : |z'| \leq |z|\}. \tag{18}$$

Now, if $|z'| \leq |z|$ and $|w(z)'| \leq |w(z)|$, we obtain $|\gamma(z')| \leq |\gamma(z)|$. By (18) we conclude that $|\gamma(z')| = |\gamma(z)|$ so that $|w(z')| = |w(z)|$. We have therefore reduced the proof that $(z, w(z))$ is minimal to the verification that $|z'| = |z|$. For $z = x \in \mathbb{R}$, $\gamma(x)$ is the unique point of $D_{|x|}$ of least modulus, and so we can conclude from $|\gamma(z')| = |\gamma(x)|$ that $\gamma(z') = \gamma(x)$. By applying γ^{-1}, we obtain that $z' = x$ so that $|z'| = |x|$, as required. The remaining case is given by $|z| = \frac{1}{\sqrt{\alpha}}$ and $z \in \mathbb{C} \setminus \mathbb{R}$. In this case, $D_{|z|}$ consists precisely of the complex numbers with modulus at least $|\gamma(z)|$, and for $|z'| < |z|$ we have $|\gamma(z')| > |\gamma(z)|$. We conclude that $|z'| = |z|$ in this case as well. $\square$

Range for t	Ordering of 0, 1, $\gamma(t)$, $\gamma(-t)$	Possible minimal point(s) obtained
$0 < t < \frac{1}{\alpha}$	$0 < \gamma(-t) < 1 < \gamma(t)$	$\{(-t, \gamma(-t))\}$
$\frac{1}{\alpha} < t < \frac{1}{\sqrt{\alpha}}$	$\gamma(t) < 0 < \gamma(-t) < 1,\ \lvert\gamma(-t)\rvert < \lvert\gamma(t)\rvert$	$\{(-t, \gamma(-t))\}$
$t = \frac{1}{\sqrt{\alpha}}$	$\gamma(t) < 0 < \gamma(-t) < 1,\ \gamma(-t) = -\gamma(t)$	$\{(z, \gamma(z)) :\ \lvert z\rvert = t,\ z \notin \mathbb{R}\}$
$\frac{1}{\sqrt{\alpha}} < t < 1$	$\gamma(t) < 0 < \gamma(-t) < 1,\ \lvert\gamma(t)\rvert < \lvert\gamma(-t)\rvert$	$\{(t, \gamma(t))\}$

TABLE 2. Possible minimal points for $0 < t < 1$.

Range for r/s	Values of x_1 and x_2
$0 < \frac{r}{s} < \mu\left(-\frac{1}{\sqrt{\alpha}}\right)$	$x_1 < -\frac{1}{\sqrt{\alpha}} < x_2 < 0$
$\frac{r}{s} = \mu\left(-\frac{1}{\sqrt{\alpha}}\right)$	$x_1 = x_2 = -\frac{1}{\sqrt{\alpha}}$
$\frac{r}{s} = \mu\left(\frac{1}{\sqrt{\alpha}}\right)$	$x_1 = x_2 = \frac{1}{\sqrt{\alpha}}$
$\frac{r}{s} > \mu\left(\frac{1}{\sqrt{\alpha}}\right)$	$\frac{1}{\alpha} < x_1 < \frac{1}{\sqrt{\alpha}} < x_2 < 1$

TABLE 3. Location of the roots of $\mu(x) = r/s$.

With the above notation, we have

$$S_{rs} = \{z \in \mathbb{C} \mid (z, \nu(z)^{-1}) \in \mathcal{M} \text{ and } f_{rs}(z) = 0\}.$$

A calculation shows that for $|z| = 1/\sqrt{\alpha}$, in order for $f_{rs}(z) = 0$, we require $z \in \mathbb{R}$. Since this case is being excluded, we may suppose that $|z| \neq \frac{1}{\sqrt{\alpha}}$. Then every minimal point has real coordinates. We wish to locate the real roots x of f_{rs} that lie in suitable intervals determined by $\mathcal{M}$. By sketching the graph of

$$\mu(x) = \frac{1}{1-x} - \frac{1}{1-\alpha x},$$

we find that for $\mu(-1/\sqrt{\alpha}) < \frac{r}{s} < \mu(1/\sqrt{\alpha})$ we have no real solutions to $\mu(x) = \frac{r}{s}$, and otherwise, we have real solutions $x_1 \leq x_2$ to $\mu(x) = \frac{r}{s}$ determined as in Table 3. Here, we have

$$\mu\left(-\frac{1}{\sqrt{\alpha}}\right) = \frac{\sqrt{\alpha}-1}{\sqrt{\alpha}+1}, \qquad \mu\left(\frac{1}{\sqrt{\alpha}}\right) = \frac{\sqrt{\alpha}+1}{\sqrt{\alpha}-1}.$$

Since $\frac{r}{s} = \mu\left(\pm\frac{1}{\sqrt{\alpha}}\right)$ results in roots having modulus $1/\sqrt{\alpha}$, this possibility has been excluded. We have therefore determined that for $\alpha > 1$ we have

$$S_{rs} = \begin{cases} \emptyset & \text{if } \frac{\sqrt{\alpha}-1}{\sqrt{\alpha}+1} \leq \frac{r}{s} \leq \frac{\sqrt{\alpha}+1}{\sqrt{\alpha}-1}; \\ \{(z_{rs}^+, \gamma(z_{rs}^+))\} & \text{otherwise.} \end{cases}$$

We note that the condition that r/s not lie in the above interval is precisely the condition that f_{rs} have distinct real roots. Replacing α with $d+1$ yields

$$r(d+1)f_{rs}(z) = (d+1)rz^2 - ((d+2)r - ds)z + r.$$

a	1	2	3	4	5
d	$1 \leq d$	$1 \leq d \leq 8$	$1 \leq d \leq 3$	1	1

TABLE 4. Values of a and d for which $(a-1)^2 d \leq 4a$.

The polynomials f_{rs} have distinct real roots $x_{rs}^+ > x_{rs}^-$ whenever

$$\frac{r}{s} \notin \left[\frac{\sqrt{d+1}-1}{\sqrt{d+1}+1}, \frac{\sqrt{d+1}+1}{\sqrt{d+1}-1}\right].$$

Putting this all together yields the following result.

Proposition 5. *With the above notation, define $x_{rs} = x_{rs}^+$. Then there exist constants $c_\ell^{(r,s)}$ for $\ell \in \mathbb{N}$ such that*

$$\sum_{k=0}^{r} (-1)^k \binom{r}{k}\binom{s}{k} d^k \sim \frac{(1-(d+1)x_{rs})^s}{r x_{rs}^r (1-x_{rs})^{s+1}} \sqrt{\frac{ds}{2\pi(d+1)\sqrt{\Delta_{f_{rs}}}}} \left(1 + \sum_{\ell=1}^{\infty} \frac{c_\ell^{(r,s)}}{s^\ell}\right)$$

as $r, s \to \infty$ $\left(\text{with } r/s,\ s/r \text{ remaining bounded and } r/s \notin \left[\frac{\sqrt{d+1}-1}{\sqrt{d+1}+1}, \frac{\sqrt{d+1}+1}{\sqrt{d+1}-1}\right]\right)$, where $\sqrt{\cdot}$ denotes the principal branch of the square root.

If we now look in the direction given by $s = ar$, Proposition 3 and Proposition 5 provide us with a proof of Theorem 1 in case $\Delta_g > 0$. We are therefore reduced to proving Theorem 1 in case $\Delta_g < 0$ and proving Theorem 2.

5. THE REMAINING CASES

The cases not covered by Section 4 all have $\varepsilon = 1$ so that our sequence of interest is given by

$$u_r = \sum_{k=0}^{r} (-1)^k \binom{r}{k}\binom{ar}{k} d^k.$$

The remaining cases correspond to $a, d \in \mathbb{N}$ such that $(a-1)^2 d \leq 4a$. These values of a and d are given in Table 4.

Recall that our plan is to calculate the polynomial $P(x,y)$ given by (11) that is satisfied by $y = F(x)$. We then use $P(x,y)$ to compute the Puiseux expansion for $F(x)$ about its dominant singularities which occur at values of x that correspond to roots z of $p(z)$. We then obtain full asymptotic expansions for u_r valid as $r \to \infty$ by applying Proposition 1. Recall further that from (7), the transfer of asymptotics for F to asymptotics for u_r can be expressed as

$$a_0(\zeta - x)^{-p/q}\left(1 + \sum_{k=1}^{\infty} c_k (\zeta - x)^k\right) \mapsto \frac{a_0 \zeta^{-p/q-r} r^{p/q-1}}{\Gamma(p/q)}\left(1 + \sum_{\ell=1}^{\infty} \frac{\mu_\ell}{r^\ell}\right). \tag{19}$$

where the constants $\mu_\ell \in \mathbb{Q}(\sqrt{\Delta_g}, c_1, \ldots, c_\ell)$. Finally, we use a linear ODE satisfied by F to obtain a linear recurrence relation satisfied by the c_k. The recurrence obtained will be used to show that all of the c_k lie in $\mathbb{Q}(\sqrt{\Delta_g})$, where g and its discriminant Δ_g are given by (3) and (4) respectively. We will then have that all of the μ_ℓ lie in $\mathbb{Q}(\sqrt{\Delta_g})$ as well. We start with the case $a = 1$.

5.1. **The case $a = 1$.** In this case, with $\alpha = d + 1$, we are considering the sequence

$$u_r = \sum_{k=0}^{r} \binom{r}{k}^2 (1-\alpha)^k,$$

which is the diagonal of the bivariate sequence given by

$$\tilde{u}_{rs} = \sum_{k=0}^{r} \binom{r}{k}\binom{s}{k}(1-\alpha)^k.$$

We find that

$$F(x) = \frac{1}{1 - 2z(x) + \alpha x},$$

where $z(x)$ is the unique root of $z(1-z) - x(1-\alpha z)$ that tends to 0 as x tends to 0. The two roots of this polynomial are given by

$$\frac{\alpha x + 1 \pm \sqrt{\alpha^2 x^2 + 2(\alpha-2)x + 1}}{2},$$

and the sign that gives the root that tends to 0 as x tends to zero is the $-$ sign. We conclude that

$$z(x) = \frac{\alpha x + 1 - \sqrt{\alpha^2 x^2 + 2(\alpha-2)x + 1}}{2},$$

so that

$$F(x) = \frac{1}{\sqrt{\alpha^2 x^2 + 2(\alpha-2)x + 1}}.$$

We see from this that the dominant singularities of F are given by the roots ζ and $\overline{\zeta}$ of

$$\alpha^2 x^2 + 2(\alpha-2)x + 1.$$

These roots are

$$\zeta = \frac{2 - \alpha - 2i\sqrt{\alpha - 1}}{\alpha^2}, \qquad \overline{\zeta} = \frac{2 - \alpha + 2i\sqrt{\alpha - 1}}{\alpha^2}.$$

We now expand $F(x)$ into a Puiseux expansion about ζ and $\overline{\zeta}$ and then transfer by way of (19) to obtain our asymptotic expansion for u_r. We find that $F(x)$ admits the following expansions in suitable neighbourhoods of ζ and $\overline{\zeta}$:

$$F(x) = a_0(\zeta - x)^{-1/2}\left(1 + \sum_{k=1}^{\infty} c_k(\zeta - x)^k\right),$$

$$F(x) = \overline{a_0}(\overline{\zeta} - x)^{-1/2}\left(1 + \sum_{k=1}^{\infty} \overline{c_k}(\overline{\zeta} - x)^k\right)$$

for constants $c_1, c_2, c_3, \ldots$ and

$$a_0 = \frac{1+i}{2^{3/2} d^{1/4}}.$$

Further, $F(x)$ satisfies the linear ODE given by

$$(\alpha + \alpha^2 x - 2)y(x) + (1 + 2\alpha x + \alpha^2 x^2 - 4x)y'(x) = 0.$$

Substituting in

$$F(x) = (\zeta - x)^{-1/2} \sum_{k=0}^{\infty} c_k(\zeta - x)^k$$

leads to the recurrence relation $c_0 = 1$ and

$$\frac{c_k}{c_{k-1}} = \frac{\alpha^2}{4\sqrt{1-\alpha}}\left(1 - \frac{1}{2k}\right) \qquad (k \geq 1).$$

We obtain

$$c_k = \frac{c_k}{c_{k-1}}\frac{c_{k-1}}{c_{k-2}}\cdots\frac{c_1}{c_0}c_0 = \frac{\alpha^{2k}}{4^k(1-\alpha)^{k/2}}\prod_{j=1}^{k}\left(1-\frac{1}{2j}\right) = \binom{k-1/2}{k}\frac{\alpha^{2k}}{4^k(1-\alpha)^{k/2}}.$$

Since each of the $c_k \in \mathbb{Q}(i\sqrt{d})$, we obtain the following result.

Proposition 6. *Let d be an integer greater than or equal to 2. There exists a decomposition*

$$\sum_{k=0}^{r}(-1)^k\binom{r}{k}^2 d^k = U_d(r) + \overline{U_d(r)}$$

where

$$U_d(r) \sim \frac{(1+i)(1-i\sqrt{d})^{2r+1}}{2^{3/2}d^{1/4}\sqrt{\pi r}}\left(1 + \sum_{\ell=1}^{\infty}\frac{\mu_\ell}{r^\ell}\right) \qquad (r \to \infty),$$

and the constants $\mu_\ell \in \mathbb{Q}(i\sqrt{d})$.

A calculation shows that this agrees with Theorem 1. Since the above calculations did not require $\varepsilon = 1$, we also obtain the following result.

Proposition 7. *Let $d \in \mathbb{N}$. There exists an asymptotic expansion*

$$\sum_{k=0}^{r}\binom{r}{k}^2 d^k \sim \frac{(\sqrt{d}+1)^{2r+1}}{2d^{1/4}\sqrt{\pi r}}\left(1 + \sum_{\ell=1}^{\infty}\frac{\mu_\ell}{r^\ell}\right) \qquad (r \to \infty),$$

where the constants $\mu_\ell \in \mathbb{Q}(\sqrt{d})$.

We now turn to the other remaining cases.

5.2. **The other cases.** Our sequence is given by

$$u_r = \sum_{k=0}^{r}(-1)^k\binom{r}{k}\binom{ar}{k}d^k.$$

The cases $2 \leq a \leq 5$ in Table 4 remain to be determined. The polynomials $P(x, y)$ given by (11) are as given in Table 5.

In case $(a, d) \in \{(2, 8), (3, 3)\}$, we have a unique dominant singularity equal to

$$\zeta = \frac{1-a}{1+a}\left(\frac{2a}{2a+(a-1)d}\right)^a.$$

Now, according to Maple, in every case we obtain only one form of a Puiseux expansion that fails to be analytic at ζ and so since we know that $F(x)$ fails to be analytic at ζ, the Puiseux expansion of F at ζ must be of this form. Further, if we use Maple to compute the Puiseux expansions of the branches of the roots of $P(x, y)$, we can conclude that the leading term of the expansion for F is off from the leading term obtained by our calculation by at worst a suitable root of unity. The correct root of unity can then be determined numerically. Also, applying the method of Frobenius to a linear ordinary differential operator with coefficients in $\mathbb{Q}[x]$ satisfied by our asymptotic series leads to a linear recurrence relation for the coefficients involved in the expansions. By checking

a	$P(x,y)$
2	$(4(d+1)^3x^2+(d^2+20d-8)x+4)y^3-((d+1)^2x+3)y-1$
3	$(27(d+1)^4x^2+(4d^3+18d^2+216d-54)x+27)y^4$ $-(3(d+3)(d+1)^2x+18)y^2-((d+1)^3x+8)y-1$
4	$(256(d+1)^5x^2+(27d^4+144d^3+320d^2+2816d-512)x+256)y^5$ $-(2(9d^2+32d+48)(d+1)^2x+160)y^3-(8(d+2)(d+1)^3x+80)y^2$ $-((d+1)^4x+15)y-1$
5	$(3125(d+1)^6x^2+(256d^5+1600d^4+4250d^3+6250d^2+43750d-6250)x+3125)y^6$ $-(10(2d+5)(8d^2+15d+25)(d+1)^2x+1875)y^4-(10(8d^2+25d+25)(d+1)^3x+1000)y^3$ $-(5(3d+5)(d+1)^4x+225)y^2-((d+1)^5x+24)y-1$

TABLE 5. The polynomials $P(x,y)$ for $2 \le a \le 5$.

sufficiently many of the terms in the sequence, this recurrence proves that all of the coefficients in question lie in $\mathbb{Q}(\sqrt{\Delta_g})$. We end up with the following propositions.

Proposition 8. *For $(a,d) \in \{(2,8),(3,3)\}$, $F(x)$ admits a Puiseux expansion of the following form in a suitable neighbourhood of $\zeta = \frac{1-a}{1+a}\left(\frac{2a}{2a+(a-1)d}\right)^a$:*

$$F(x) = \frac{a_0}{(\zeta-x)^{2/3}}\left(1+\sum_{k=1}^{\infty} c_k(\zeta-x)^k\right) + \frac{b_0}{(\zeta-x)^{1/3}}\left(1+\sum_{k=1}^{\infty} d_k(\zeta-x)^k\right)$$

in case $(a,d)=(2,8)$ and

$$F(x) = \frac{a_0}{(\zeta-x)^{2/3}}\left(1+\sum_{k=1}^{\infty} c_k(\zeta-x)^k\right) + \frac{b_0}{(\zeta-x)^{1/3}}\left(1+\sum_{k=1}^{\infty} d_k(\zeta-x)^k\right) + \sum_{k=0}^{\infty} e_k(\zeta-x)^k$$

in case $(a,d)=(3,3)$. Here, the constants c_k and d_k lie in $\mathbb{Q}$.

Proposition 9. *Suppose that*

$$(a,d) \in \{(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),(2,7),(3,1),(3,2),(4,1),(5,1)\}.$$

Then, with the above notation, $F(x)$ admits a Puiseux expansion of the following form in suitable neighbourhoods of ζ and $\overline{\zeta}$ respectively:

$$F(x) = \frac{a_0}{\sqrt{\zeta-x}}\left(1+\sum_{k=1}^{\infty} c_k(\zeta-x)^k\right) + \sum_{k=0}^{\infty} b_k(\zeta-x)^k$$

and

$$F(x) = \frac{\overline{a_0}}{\sqrt{\overline{\zeta}-x}}\left(1+\sum_{k=1}^{\infty} \overline{c_k}(\overline{\zeta}-x)^k\right) + \sum_{k=0}^{\infty} \overline{b_k}(\overline{\zeta}-x)^k$$

where each of the c_k lies in $\mathbb{Q}(\sqrt{\Delta_g})$.

Using the transfer method of Flajolet and Sedgewick, we obtain the following asymptotics for our sequence u_r.

Proposition 10. *Let* $(a,d) \in \{(2,8),(3,3)\}$, $\zeta = \frac{1-a}{1+a}\left(\frac{2a}{2a+(a-1)d}\right)^a$, *and*

$$u_r = \sum_{k=0}^{r}(-1)^k \binom{r}{k}\binom{ar}{k} d^k.$$

There exist constants a_0, b_0 *and* $\mu_\ell, \eta_\ell \in \mathbb{Q}$ *for* $\ell \geq 1$ *such that*

$$u_r \sim \frac{a_0 \zeta^{-r}}{\Gamma(2/3)\zeta^{2/3} r^{1/3}}\left(1 + \sum_{\ell=1}^{\infty} \frac{\mu_\ell}{r^\ell}\right) + \frac{b_0 \zeta^{-r}}{\Gamma(1/3)\zeta^{1/3} r^{2/3}}\left(1 + \sum_{\ell=1}^{\infty} \frac{\eta_\ell}{r^\ell}\right) \qquad (r \to \infty)$$

Proposition 11. *Suppose that*

$$(a,d) \in \{(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),(2,7),(3,1),(3,2),(4,1),(5,1)\}.$$

Then, with the above notation, we have a decomposition

$$\sum_{k=0}^{r}(-1)^k \binom{r}{k}\binom{ar}{k} d^k = U(r) + \overline{U(r)}$$

for which

$$U(r) \sim \frac{a_0 \zeta^{-r}}{\sqrt{\pi \zeta r}}\left(1 + \sum_{\ell=1}^{\infty} \frac{\mu_\ell}{r^\ell}\right) \qquad (r \to \infty)$$

for some constants $\mu_\ell \in \mathbb{Q}(\sqrt{\Delta_g})$.

In each case, a calculation using Maple shows that we obtain the same leading term as is given in Theorem 1 and Theorem 2. We have therefore completed the proof of our main result.

6. Examples

Having proved our main result, we now conclude this paper with some examples.

Example 1. In the limiting case $\varepsilon = 0$ and $d \to 1^+$ we obtain the asymptotic expansion of the binomial coefficients given by Stirling's formula. Let $a \in \mathbb{N}$. There exist constants $\mu_\ell(a)$ for $\ell \in \mathbb{N}$ such that

$$\binom{(a+1)r}{r} = \sum_{k=0}^{r}\binom{r}{k}\binom{ar}{k} \sim \frac{\delta \beta^r}{\sqrt{2\pi r}}\left(1 + \sum_{\ell=1}^{\infty} \frac{\mu_\ell(a)}{r^\ell}\right) \qquad (r \to \infty)$$

where

$$\delta = \sqrt{\frac{a+1}{a}}, \qquad \beta = \frac{(a+1)^{a+1}}{a^a}.$$

In particular, the central binomial coefficients satisfy

$$\binom{2r}{r} \sim \frac{4^r}{\sqrt{\pi r}}\left(1 + \sum_{\ell=1}^{\infty} \frac{\mu_\ell(1)}{r^\ell}\right) \qquad (r \to \infty),$$

and the Catalan numbers satisfy

$$\frac{1}{r+1}\binom{2r}{r} \sim \frac{4^r}{(r+1)\sqrt{\pi r}}\left(1 + \sum_{\ell=1}^{\infty} \frac{\mu_\ell(1)}{r^\ell}\right) \qquad (r \to \infty).$$

Example 2. Proposition 7 provides us with an asymptotic expansion for generalizations of the central Delannoy numbers. For $d \in \mathbb{N}$ we have constants $\mu_\ell(d) \in \mathbb{Q}(\sqrt{d})$ such that

$$\sum_{k=0}^{r} \binom{r}{k}^2 d^k \sim \frac{(\sqrt{d}+1)^{2r+1}}{2d^{1/4}\sqrt{\pi r}} \left(1 + \sum_{\ell=1}^{\infty} \frac{\mu_\ell(d)}{r^\ell}\right) \qquad (r \to \infty).$$

In particular, the central Delannoy numbers satisfy

$$\sum_{k=0}^{r} \binom{r}{k}^2 2^k \sim \frac{(2^{1/4}+2^{-1/4})}{2\sqrt{\pi r}} (3+2\sqrt{2})^r \left(1 + \sum_{\ell=1}^{\infty} \frac{\mu_\ell(2)}{r^\ell}\right) \qquad (r \to \infty),$$

where the $\mu_\ell(2)$ lie in $\mathbb{Q}(\sqrt{2})$.

Example 3. Proposition 6 provides us with an asymptotic expansion for generalizations of the alternating analogue of the central Delannoy numbers. Let Let d be an integer greater than or equal to 2. There exists a decomposition

$$\sum_{k=0}^{r} (-1)^k \binom{r}{k}^2 d^k = U_d(r) + \overline{U_d(r)}$$

where

$$U_d(r) \sim \frac{(1+i)(1-i\sqrt{d})^{2r+1}}{2^{3/2}d^{1/4}\sqrt{\pi r}} \left(1 + \sum_{\ell=1}^{\infty} \frac{\mu_\ell(d)}{r^\ell}\right) \qquad (r \to \infty),$$

for constants $\mu_\ell(d) \in \mathbb{Q}(i\sqrt{d})$. In particular, the alternating analogue of the central Delannoy numbers satisfies

$$\sum_{k=0}^{r} (-1)^k \binom{r}{k}^2 2^k = U_2(r) + \overline{U_2(r)}$$

where

$$U_2(r) \sim \frac{(1+i)(1-i\sqrt{2})^{2r+1}}{2^{7/4}\sqrt{\pi r}} \left(1 + \sum_{\ell=1}^{\infty} \frac{\mu_\ell(2)}{r^\ell}\right) \qquad (r \to \infty),$$

for constants $\mu_\ell(2) \in \mathbb{Q}(i\sqrt{2})$.

Example 4 (The Conjecture of Chamberland and Dilcher)**.** The special case given by $\varepsilon = 1$, $a = 2$, $d = 1$ yields

$$\sum_{k=0}^{r} (-1)^k \binom{r}{k}\binom{2r}{k} = U(r) + \overline{U(r)},$$

$$U(r) \sim \frac{\delta \beta^r}{\sqrt{2\pi r}} \left(1 + \sum_{\ell=1}^{\infty} \frac{\mu_\ell}{r^\ell}\right) \qquad (r \to \infty)$$

where

$$\delta = \frac{1}{\sqrt{-\sqrt{-7}}} \left(\frac{-31-3\sqrt{-7}}{8}\right)^{1/4}, \qquad \beta = \frac{-13+7\sqrt{-7}}{8}$$

and the μ_ℓ lie in $\mathbb{Q}(\sqrt{-7})$. In [2] the authors conjecture that the coefficient of $\beta^r/\sqrt{r}$ is very close to

$$0.3468 \exp\left(i\pi \frac{20}{1001}\right) \approx .3461170356 + 0.02175402677i.$$

The correct value of this coefficient evaluates to

$$\frac{\delta}{\sqrt{2\pi}} = \frac{1}{\sqrt{-2\pi\sqrt{-7}}}\left(\frac{-31-3\sqrt{-7}}{8}\right)^{1/4} \approx .3461762814 + 0.02172120012i.$$

References

[1] George D. Birkhoff and W. J. Trjitzinsky. Analytic theory of singular difference equations. *Acta Math.*, 60(1):1–89, 1933.

[2] Marc Chamberland and Karl Dilcher. A binomial sum related to Wolstenholme's theorem. *J. Number Theory*, 129(11):2659–2672, 2009.

[3] Earl A. Coddington and Norman Levinson. *Theory of ordinary differential equations*. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1955.

[4] Louis Comtet. *Advanced combinatorics*. D. Reidel Publishing Co., Dordrecht, enlarged edition, 1974. The art of finite and infinite expansions.

[5] Philippe Flajolet and Robert Sedgewick. *Analytic combinatorics*. Cambridge University Press, Cambridge, 2009.

[6] Harry Furstenberg. Algebraic functions over finite fields. *J. Algebra*, 7:271–277, 1967.

[7] I. M. Gel′fand, M. M. Kapranov, and A. V. Zelevinsky. *Discriminants, resultants, and multidimensional determinants*. Mathematics: Theory & Applications. Birkhäuser Boston Inc., Boston, MA, 1994.

[8] Donald E. Knuth. *The art of computer programming. Volume 3*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1973. Sorting and searching, Addison-Wesley Series in Computer Science and Information Processing.

[9] Maple. `http://www.maplesoft.com`.

[10] Richard J. McIntosh. An asymptotic formula for binomial sums. *J. Number Theory*, 58(1):158–172, 1996.

[11] A. M. Odlyzko. Asymptotic enumeration methods. In *Handbook of combinatorics, Vol. 1, 2*, pages 1063–1229. Elsevier, Amsterdam, 1995.

[12] Robin Pemantle and Mark C. Wilson. Asymptotics of multivariate sequences. I. Smooth points of the singular variety. *J. Combin. Theory Ser. A*, 97(1):129–161, 2002.

[13] Richard P. Stanley. *Enumerative combinatorics. Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin.

[14] Michael Stoll and Bruno Haible. Asymptotic expansions of P-recursive sequences. Preprint, July 1994.

[15] Mark C. Wilson. Asymptotics for generalized Riordan arrays. In *2005 International Conference on Analysis of Algorithms*, Discrete Math. Theor. Comput. Sci. Proc., AD, pages 323–333 (electronic). Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2005.

[16] Jet Wimp. Current trends in asymptotics: some problems and some solutions. In *Proceedings of the Fourth International Congress on Computational and Applied Mathematics (Leuven, 1990)*, volume 35, pages 53–79, 1991.

[17] Jet Wimp and Doron Zeilberger. Resurrecting the asymptotics of linear recurrences. *J. Math. Anal. Appl.*, 111(1):162–176, 1985.

Department of Mathematics and Statistics, Dalhousie University, Halifax, NS, Canada, B3H 3J5