Asymptotics of a family of binomial sums

Rob Noble

Department of Mathematics and Statistics
Dalhousie University

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Outline

The problem

Conjecture (Chamberland and Dilcher)

The sequence $a_r = \sum_{k=0}^{r}(-1)^k\binom{r}{k}\binom{2r}{k}$ *satisfies*

$$a_r \sim \frac{d\alpha^r}{\sqrt{r}}\left(1+\frac{c_1}{r}+\frac{c_2}{r^2}+\dots\right)+\frac{\overline{d\alpha}^r}{\sqrt{r}}\left(1+\frac{\overline{c_1}}{r}+\frac{\overline{c_2}}{r^2}+\dots\right)$$

as $r\to\infty$*, for some* $d, c_1, c_2, \dots \in \mathbb{C}$*, where* $\alpha = \frac{-13+i7\sqrt{7}}{8}$.

- Problem: Study the asymptotics of the binomial sums

$$u_r^{(\varepsilon,a,d)} = \sum_{k=0}^{r}(-1)^{\varepsilon k}\binom{r}{k}\binom{ar}{k}d^k,$$

for $\varepsilon \in \{0,1\}$ and $a, d \in \mathbb{N}$ as $r\to\infty$.

Examples

- Central binomial coefficients:

$$\binom{2r}{r} = u_r^{(0,1,1)} = \sum_{k=0}^{r} \binom{r}{k}^2;$$

- Central Delannoy numbers:

$$D(r,r) = u_r^{(0,1,2)} = \sum_{k=0}^{r} \binom{r}{k}^2 2^k;$$

- Binomial sum considered by Chamberland and Dilcher:

$$u_r^{(1,2,1)} = \sum_{k=0}^{r} (-1)^k \binom{r}{k}\binom{2r}{k}.$$

Results: Case I

- $\alpha = 1 - (-1)^{\varepsilon} d$,
- $a(\alpha - 1)g(z) = \alpha z^2 + (a\alpha - a - \alpha - 1)z + 1$.
- Δ_g - the discriminant of g
- z_0 - A particularly chosen root of g.
- $\delta = \frac{1}{(1-z_0)\sqrt[4]{\Delta_g}}, \quad \beta = \frac{1}{z_0}\left(\frac{1-\alpha z_0}{1-z_0}\right)^a$

Theorem ($\Delta_g > 0$ Case)

There exist constants μ_ℓ such that

$$\sum_{k=0}^{r} (-1)^{\varepsilon k} \binom{r}{k} \binom{ar}{k} d^k \sim \frac{\delta \beta^r}{\sqrt{2\pi r}} \left(1 + \sum_{\ell=1}^{\infty} \frac{\mu_\ell}{r^\ell}\right) \qquad (r \to \infty)$$

Results: Case II

- $\alpha = 1 - (-1)^{\varepsilon} d$,
- $a(\alpha - 1)g(z) = \alpha z^2 + (a\alpha - a - \alpha - 1)z + 1$.
- Δ_g - the discriminant of g
- z_0 - A particularly chosen root of g.
- $\delta = \frac{1}{(1-z_0)\sqrt[4]{\Delta_g}}, \quad \beta = \frac{1}{z_0}\left(\frac{1-\alpha z_0}{1-z_0}\right)^a$

Theorem ($\Delta_g < 0$ Case)

There exist constants μ_ℓ such that, as $r \to \infty$,

$$\sum_{k=0}^{r} (-1)^{\varepsilon k} \binom{r}{k} \binom{ar}{k} d^k \sim \frac{\delta \beta^r}{\sqrt{2\pi r}} \left(1 + \sum_{\ell=1}^{\infty} \frac{\mu_\ell}{r^\ell}\right) + \frac{\overline{\delta\beta}^r}{\sqrt{2\pi r}} \left(1 + \sum_{\ell=1}^{\infty} \frac{\overline{\mu_\ell}}{r^\ell}\right).$$

Results: Case III (1 of 2)

Theorem ($\Delta_g = 0$ Case)

There exist constants $\mu_\ell, \eta_\ell \in \mathbb{Q}$ with denominators divisible only by the primes 2 and 3 such that

$$\sum_{k=0}^{r} (-1)^k \binom{r}{k}\binom{2r}{k} 8^k \sim \frac{(-27)^r}{2^{2/3}\Gamma(2/3) r^{1/3}} \left(1 + \sum_{\ell=1}^{\infty} \frac{\mu_\ell}{r^\ell}\right) + \frac{(-27)^r}{2^{4/3}\Gamma(1/3) r^{2/3}} \left(1 + \sum_{\ell=1}^{\infty} \frac{\eta_\ell}{r^\ell}\right) \qquad (r \to \infty).$$

Results: Case III (2 of 2)

Theorem ($\Delta_g = 0$ Case)

There exist constants $\tilde{\mu}_\ell, \tilde{\eta}_\ell \in \mathbb{Q}$ such that

$$\sum_{k=0}^{r} (-1)^k \binom{r}{k} \binom{3r}{k} 3^k \sim \frac{2^{2/3}(-16)^r}{3\Gamma(2/3) r^{1/3}} \left(1 + \sum_{\ell=1}^{\infty} \frac{\tilde{\mu}_\ell}{r^\ell}\right) + \frac{2^{1/3}(-16)^r}{3\Gamma(1/3) r^{2/3}} \left(1 + \sum_{\ell=1}^{\infty} \frac{\tilde{\eta}_\ell}{r^\ell}\right) \qquad (r \to \infty).$$

Generalized Riordan Arrays

Definition

$\{a_{rs}\}_{r,s}$ is a *generalized Riordan array* if

$$\tilde{F}(z,w) = \sum_{r,s \geqslant 0} a_{rs} z^r w^s = \frac{\varphi(z)}{1 - w\nu(z)}$$

for meromorphic functions φ and ν that are analytic at $z = 0$.

Lemma

For $\tilde{u}_{rs} = \sum_{k=0}^{r} (-1)^{\varepsilon k} \binom{r}{k} \binom{as}{k} d^k = \sum_{k=0}^{r} \binom{r}{k} \binom{as}{k} (1-\alpha)^k$ *we can take*

$$\varphi(z) = \frac{1}{1-z}, \qquad \nu(z) = \left(\frac{1-\alpha z}{1-z}\right)^a.$$

Main result of Pemantle and Wilson used

- $\tilde{F}(z,w) = \sum_{r,s\geqslant 0} a_{rs} z^r w^s = \frac{\varphi(z)}{1-w\nu(z)}$, $Q_{rs}(z) = \frac{z^2\nu''(z)}{\nu(z)} - \frac{r(r-s)}{s^2}$.
- A pole (z,w) is minimal if for every pole (z',w'), $|z'| \leqslant |z|$ and $|w'| \leqslant |w|$ imply $|z'| = |z|$ and $|w'| = |w|$.
- $S_{rs} = \{z \in \mathbb{C} \mid (z, \nu(z)^{-1})$ is minimal , $\varphi(z) \neq 0, sz\nu'(z) = r\nu(z)$, and $sz\nu''(z) \neq (r-s)\nu'(z)\}$.

Proposition (Pemantle and Wilson)

Under suitable conditions and if S_{rs} is finite and nonempty then there exists an asymptotic expansion

$$a_{rs} \sim \sum_{z_{rs}\in S_{rs}} \frac{\varphi(z_{rs})\nu(z_{rs})^s}{z_{rs}^r\sqrt{2\pi s Q_{rs}(z_{rs})}} \left(1 + \sum_{\ell=1}^{\infty} \frac{c_\ell^{(z_{rs})}}{s^\ell}\right)$$

as $r,s \to \infty$ (with r/s, s/r remaining bounded), where $\sqrt{\cdot}$ denotes the principal branch of the square root.

Classifying the set S_{rs}

In our case:

- $S_{rs} = \{z \in \mathbb{C} \mid (z, \nu(z)^{-1})$ is minimal and $r\alpha z^2 - ((1+\alpha)r + (1-\alpha)s)z + r = 0\}$.
- $\nu(z)^{-1} = \left(\frac{1-z}{1-\alpha z}\right)^a = \gamma(z)^a$ for $\gamma(z) = \frac{1-z}{1-\alpha z}$.
- Using the fact that Möbius transformations send circles to circles we can classify the minimal points.
- We end up being able to apply the result of Pemantle and Wilson in all but finitely many cases.

Dealing with the remaining cases

- Find the singularities of the generating function having least nonzero modulus.
  - The generating function satisfies parametric equations and we can differentiate implicitly to find the singularities.
- Expand the generating function about these singularities.
  - A linear homogeneous ODE with polynomial coefficients satisfied by the generating function can be used for this.
  - An algebraic equation satisfied by the generating function can also be used for this.
- Transfer the asymptotic expansion of the generating function over to its coefficient sequence by the transfer method of Flajolet and Sedgewick.

The main result of Flajolet and Sedgewick

Proposition (Flajolet, Sedgewick)

Suppose that $\zeta_1, \ldots, \zeta_n$ are the dominant singularities of the ordinary generating function F of the sequence $\{a_r\}_r$. Under certain conditions if F admits an expansion of the form

$$F(z) \sim \sum_{k \geqslant k_j} c_{j,k} (\zeta_j - z)^{k-\theta} \qquad (z \to \zeta_j),$$

for all j then

$$a_r \sim \sum_{j=1}^{n} \frac{c_{j,k_j} r^{\theta-1} \zeta_j^{k_j-\theta-r}}{\Gamma(\theta - k_j)} \left(1 + \sum_{\ell=k_j+1}^{\infty} \frac{\mu_{j,\ell}}{r^\ell} \right) \qquad (r \to \infty)$$

where $\mu_{j,\ell} \in \mathbb{Q}(\theta, \zeta_j, c_{j,k_j+1}/c_{j,k_j}, \ldots, c_{j,\ell}/c_{j,k_j})$ for each j and ℓ.

Divisibility properties of the asymptotic coefficients

- Stoll and Haible developed a method that can find divisibility properties of the asymptotic coefficients in certain asymptotic expansions.
- Applying the method we obtain the following result.

Proposition

Let $d \in \mathbb{N}$. There exists an asymptotic expansion

$$\sum_{k=0}^{r} \binom{r}{k}^2 d^k \sim \frac{(\sqrt{d}+1)^{2r+1}}{2d^{1/4}\sqrt{\pi r}} \left(1 + \sum_{\ell=1}^{\infty} \frac{\mu_\ell}{r^\ell}\right) \qquad (r \to \infty),$$

where the constants $\mu_\ell \in \mathbb{Q}(\sqrt{d})$ are such that the only primes of $\mathbb{Q}(\sqrt{d})$ that can divide their denominators are the prime divisors of 2 and the prime divisors of $\sqrt{d}$.

Summary

- The sequences $\sum_{k=0}^{r}(-1)^{\varepsilon k}\binom{r}{k}\binom{ar}{k}d^k$ for $\varepsilon \in \{0,1\}$ and $a, d \in \mathbb{N}$ admit full asymptotic expansions as $r \to \infty$.
- The main terms can be given explicitly in all cases.
- On an individual basis, a field containing the asymptotic coefficients can be found, and for the case $\varepsilon = 0$ and $a = 1$, the divisibility properties of the asymptotic coefficients can be found.
- Open Questions.
  - Can a number field containing the asymptotic coefficients be found in general?
  - Can the divisibility properties of the asymptotic coefficients be found in general?