Asymptotics of the weighted Delannoy numbers

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Outline

An example

Example (Central Delannoy numbers $D(r,r)$)

- $D(r,r)$: $\#$ of paths from $(0,0)$ to (r,r) using steps $(1,0)$, $(0,1)$, $(1,1)$.
- $D(r,r) = \sum_{k=0}^{r} \binom{r}{k}^2 2^k$.
- Asymptotic expansion: As $r \to \infty$,

$$D(r,r) \sim \frac{(3+2\sqrt{2})^r}{2\sqrt{(3\sqrt{2}-4)\pi r}} \left(1 - \frac{8-3\sqrt{2}}{32r} + \frac{113-72\sqrt{2}}{1024r^2} + \ldots\right).$$

- First few coefficients equal to an element of $\mathbb{Z}[\sqrt{2}]$ divided by a power of 2.
- Factored over $\mathbb{Q}(\sqrt{2})$: only prime dividing denominators of first few is $\sqrt{2}\mathbb{Z}[\sqrt{2}]$.
- Pattern continues: only prime of $\mathbb{Q}(\sqrt{2})$ dividing denominators is $\sqrt{2}\mathbb{Z}[\sqrt{2}]$.

The weighted Delannoy numbers

- $\alpha, \beta, \gamma \in \mathbb{C}$, $r, s \in \mathbb{N}_0$.
- Paths $(0,0)$ to (r,s) using steps:

$$(1,0) \text{ with weight } \alpha, \quad (0,1) \text{ with weight } \beta, \quad (1,1) \text{ with weight } \gamma.$$

- Weight of path $=$ product of weights of steps in path.
- Weighted Delannoy numbers $u_{r,s} =$ total of weights of paths to (r,s).
- Recurrence relation: $u_{r,0} = \alpha^r$ $(r \geqslant 0)$, $u_{0,s} = \beta^s$ $(s \geqslant 0)$,

$$u_{r+1,s+1} = \alpha u_{r,s+1} + \beta u_{r+1,s} + \gamma u_{r,s} \qquad (r,s \geqslant 0).$$

- Formula:

$$u_{r,s} = \sum_{k=0}^{r} \binom{r}{k} \binom{s}{k} \alpha^{r-k} \beta^{s-k} (\alpha\beta + \gamma)^k$$

- $\alpha = \beta = \gamma = 1$, $r = s$ gives $D(r,r)$.

Denominators of algebraic numbers

- K a number field
- $\mathcal{O}_K$ its ring of integers
- $\delta \in K^*$. We have

$$\delta\mathcal{O}_K = \prod_{\mathfrak{p}} \mathfrak{p}^{v_{\mathfrak{p}}(\delta)}$$

  $\mathfrak{p}$ a nonzero prime ideal of $\mathcal{O}_K$, $v_{\mathfrak{p}}(\delta) \in \mathbb{Z}$ all but finitely many equal to zero.
- Prime divisors of δ: $\mathfrak{p}$ for which $v_{\mathfrak{p}}(\delta) > 0$
- Denominator of δ: product of $\mathfrak{p}^{-v_{\mathfrak{p}}(\delta)}$ over primes $\mathfrak{p}$ for which $v_{\mathfrak{p}}(\delta) < 0$.

- $d = 1 + \frac{\gamma}{\alpha\beta} \in \mathbb{R}$, $u_{r,r} = \alpha^r \beta^r \sum_{k=0}^{r} \binom{r}{k}^2 d^k$.

Proposition

- *As $r \to \infty$:*
  - *$d > 0$:*

$$u_{r,r} \sim \alpha^r \beta^r \frac{(1+\sqrt{d})^{2r+1}}{2\sqrt[4]{d}\sqrt{\pi r}} \left(1 + \sum_{\ell=1}^{\infty} \frac{\mu_\ell}{r^\ell}\right)$$

  - *$d < 0$:*

$$u_{r,r} \sim \alpha^r \beta^r \frac{(1+\sqrt{d})^{2r+1}}{2\sqrt[4]{d}\sqrt{\pi r}} \left(1 + \sum_{\ell=1}^{\infty} \frac{\mu_\ell}{r^\ell}\right) + \alpha^r \beta^r \frac{(1-\sqrt{d})^{2r+1}}{2\overline{\sqrt[4]{d}}\sqrt{\pi r}} \left(1 + \sum_{\ell=1}^{\infty} \frac{\overline{\mu_\ell}}{r^\ell}\right)$$

- *$\mu_\ell \in \mathbb{Q}(\sqrt{d})$.*
- *d algebraic $\implies$ only primes of $\mathbb{Q}(\sqrt{d})$ that can divide denominators of μ_ℓ are prime divisors of 2 and $\sqrt{d}$.*

Results (2 of 2)

- $$\gamma = -9\alpha\beta,$$
- $$u_{r,2r} = \alpha^r \beta^{2r} \sum_{k=0}^{r} (-1)^k \binom{r}{k} \binom{2r}{k} 8^k.$$

Proposition

There exist $\mu_\ell, \eta_\ell \in \mathbb{Q}$ for $\ell \in \mathbb{N}$, the denominators of which are divisible only by the primes 2 and 3 such that, as $r \to \infty$,

$$u_{r,2r} \sim \frac{(-27\alpha\beta^2)^r}{2^{2/3}\Gamma(2/3)r^{1/3}} \left(1 + \sum_{\ell=1}^{\infty} \frac{\mu_\ell}{r^\ell}\right) + \frac{(-27\alpha\beta^2)^r}{2^{4/3}\Gamma(1/3)r^{2/3}} \left(1 + \sum_{\ell=1}^{\infty} \frac{\eta_\ell}{r^\ell}\right).$$

Existence of the asymptotic expansion (1 of 2)

- Find the singularities of the generating function having least nonzero modulus.
  - The generating function satisfies parametric equations and we can differentiate implicitly to find the singularities.
- Expand the generating function about these singularities.
  - A linear homogeneous ODE with polynomial coefficients satisfied by the generating function can be used for this.
  - An algebraic equation satisfied by the generating function can also be used for this.
- Transfer the asymptotic expansion of the generating function over to its coefficient sequence by the transfer method of Flajolet and Sedgewick.

Existence of the asymptotic expansion (2 of 2)

Proposition (Flajolet, Sedgewick)

$\zeta_1, \ldots, \zeta_n$: dominant singularities of the generating function F of the sequence $\{a_r\}_r$. Under certain conditions if F admits an expansion of the form

$$F(z) \sim \sum_{k \geqslant k_j} c_{j,k}(\zeta_j - z)^{k-\theta} \qquad (z \to \zeta_j),$$

for all j then

$$a_r \sim \sum_{j=1}^{n} \frac{c_{j,k_j} r^{\theta-1} \zeta_j^{k_j-\theta-r}}{\Gamma(\theta - k_j)} \left(1 + \sum_{\ell=k_j+1}^{\infty} \frac{\mu_{j,\ell}}{r^\ell}\right) \qquad (r \to \infty)$$

where $\mu_{j,\ell} \in \mathbb{Q}(\theta, \zeta_j, c_{j,k_j+1}/c_{j,k_j}, \ldots, c_{j,\ell}/c_{j,k_j})$ for each j and ℓ.

Divisibility properties of the coefficients

- $\varphi \in \mathbb{Q}$, $q \in \mathbb{N}$, $F(x) = \sum_{r=0}^{\infty} f_r x^r \in \mathbb{C}[\![x]\!]$

$$f_r \sim r^{-\varphi} \sum_{m=N}^{\infty} \frac{a_m}{r^{m/q}} \qquad (r \to \infty)$$

  for some integer N and sequence $\{a_m\}_{m \geqslant N} \subseteq \mathbb{C}$

- $$\Psi(F) = \sum_{k=N}^{\infty} \frac{a_k}{\Gamma(\varphi + k/q)} \log(1+x)^{\varphi + k/q - 1} \in x^{\varphi}\mathbb{C}(\!(x^{1/q})\!).$$

Divisibility properties of the coefficients

- $F(x) = \sum_{r=0}^{\infty} f_r x^r$,

$$f_r \sim r^{-\varphi} \sum_{m=N}^{\infty} \frac{a_m}{r^{m/q}} \qquad (r \to \infty)$$

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$$\Psi(F(x)) = \frac{x^{\varphi+N/q-1}}{\Gamma(\varphi + N/q)} \sum_{n=0}^{\infty} b_n x^{n/q}$$

Proposition

Let K be a number field and $0 \leqslant \ell < q$.

- $b_\ell \neq 0 \implies a_{N+\ell} \neq 0$. *In this case,*
  - $b_{qn+\ell}/b_\ell \in K$ *for all* $n \implies a_{qk+N+\ell}/a_{N+\ell} \in K$ *for all* k, *and*
  - *only primes of* K *that can divide the denominator of* $a_{qk+N+\ell}/a_{N+\ell}$ *are the primes that divide the denominator of* $\varphi + N/q + \ell/q$ *or* $n!b_{qn+\ell}/b_\ell$ *for some* $0 \leqslant n \leqslant k$.

Divisibility properties of the coefficients

- $F(x) = \sum_{r=0}^{\infty} f_r x^r, \qquad f_r \sim r^{-\varphi} \sum_{m=N}^{\infty} \frac{a_m}{r^{m/q}} \quad (r \to \infty)$

Corollary

Suppose:

- *K is a number field.*
- *$\{b_n\}_n$ is defined by*

$$\Psi(F(x)) = \frac{1}{\sqrt{\pi x}} \left(1 + \sum_{n=1}^{\infty} b_n x^n \right),$$

where each $b_n \in K$.

Then:

- *$a_k \in K$ for $k \geqslant 0$ and,*
- *only primes that can divide their denominators are the primes dividing 2 or the denominator of some $n! b_n$.*

Divisibility properties of the coefficients

- $F(x) = \sum_{r=0}^{\infty} f_r x^r, \qquad f_r \sim r^{-\varphi} \sum_{m=N}^{\infty} \frac{a_m}{r^{m/q}} \quad (r \to \infty)$

Corollary

Suppose:

- *K is a number field.*
- *$\{c_n\}_n$ and $\{d_n\}_n$ are defined by*

$$\Psi(F(x)) = \frac{x^{-2/3}}{\Gamma(1/3)} \left(1 + \sum_{n=1}^{\infty} c_n x^n\right) + b x^{-1/3} \left(1 + \sum_{n=1}^{\infty} d_n x^n\right),$$

where each $c_n, d_n \in K$ and $b \in \mathbb{C}$ is nonzero.

Then:

- *$a_{3k}, a_{3k+1}/a_1 \in K$ for $k \geqslant 0$ and,*
- *only primes that can divide their denominators are the primes dividing 3 or the denominator of some $n!c_n$ or the denominator of some $n!d_n$.*

Divisibility properties of the coefficients

- F: generating function of

$$f_r := \frac{2\sqrt[4]{d}\sqrt{\pi}u_{r,r}}{\alpha^r\beta^r(\sqrt{d}+1)^{2r+1}}.$$

- F satisfies the following ODE:

$$((1-\sqrt{d})^2x^2-2(1+d)x+(1+\sqrt{d})^2)F'(x)+((1-\sqrt{d})^2x-(1+d))F(x)=0.$$

- $B(x)=\Psi(F(x))$ satisfies the ODE

$$((1-\sqrt{d})^2(x+1)^2-2(1+d)(x+1)+(1+\sqrt{d})^2)B'(x)$$
$$+((1-\sqrt{d})^2(x+1)-(1+d))B(x)=0.$$

- Solving for B we obtain, for some constant C

$$B(x)=\frac{C}{\sqrt{x(4\sqrt{d}-(\sqrt{d}-1)^2x)}}$$

Divisibility properties of the coefficients

- But
$$B(x) = \frac{1}{\sqrt{\pi x}}\left(1 + \sum_{r=1}^{\infty} b_r x^r\right). \tag{1}$$

- Implies
$$B(x) = \frac{2\sqrt[4]{d}}{\sqrt{\pi x}\sqrt{4\sqrt{d} - (\sqrt{d}-1)^2 x}} = \frac{1}{\sqrt{\pi x}}\left(1 - \frac{(\sqrt{d}-1)^2}{4\sqrt{d}}x\right)^{-1/2}. \tag{2}$$

- Comparing the right-hand sides of (1) and (2) yields
$$b_r = \binom{-1/2}{r}(-1)^r\delta^r = \binom{r-1/2}{r}\delta^r \qquad \text{where} \qquad \delta = \frac{(\sqrt{d}-1)^2}{4\sqrt{d}}.$$

- Implies only primes of $\mathbb{Q}(\sqrt{d})$ that can divide the denominators of coefficients are the prime divisors of 2 and $\sqrt{d}$.

Divisibility properties of the coefficients

- F: generating function of $\{f_r\}_r$ given by

$$f_r := \frac{2^{2/3}\Gamma(2/3)u_{r,2r}}{(-27)^r\alpha^r\beta^{2r}}$$

- F satisfies ODE

$$(18x^3 - 36x^2 + 18x)F''(x) + (45x^2 - 54x + 9)F'(x) + (9x - 5)F(x) = 0.$$

- $B(x) := \Psi(F(x))$ satisfies ODE

$$18x^2(x+1)B''(x) + 9x(5x+4)B'(x) + (9x+4)B(x) = 0.$$

- Solve: for some constants C_1, C_2:

$$B(x) = C_1 x^{-2/3}{}_2F_1\left(-\frac{1}{6}, \frac{1}{3}; \frac{2}{3}; -x\right) + C_2 x^{-1/3}{}_2F_1\left(\frac{1}{6}, \frac{2}{3}; \frac{4}{3}; -x\right)$$

Divisibility properties of the coefficients

- But

$$\Psi(F(x)) = \frac{x^{-2/3}}{\Gamma(1/3)}\left(1+\sum_{n=1}^{\infty} c_n x^n\right) + \frac{x^{-1/3}}{2^{2/3}\Gamma(1/3)}\left(1+\sum_{n=1}^{\infty} d_n x^n\right)$$

  for certain c_n and d_n

- Implies

$$c_n = [x^n]_2F_1\left(-\frac{1}{6},\frac{1}{3};\frac{2}{3};-x\right) = \frac{(-1)^n(-1/6)_n(1/3)_n}{n!(2/3)_n},$$
$$d_n = [x^n]_2F_1\left(\frac{1}{6},\frac{2}{3};\frac{4}{3};-x\right) = \frac{(-1)^n(1/6)_n(2/3)_n}{n!(4/3)_n},$$

  where $(t)_n = t(t+1)\ldots(t+n-1)$ denotes the rising Pochhammer symbol.

- Implies the only primes that can divide the denominators of the coefficients are 2 and 3.

Thank You.