What is Continuity? Why Should I Care?

The word "continuous" occurs often in common parlance and refers to anything that has no breaks or gaps, as in "It has been raining continuously for two hours." In mathematics, however, "continuous" and "continuity" have a precise technical meaning, which, while it includes the common sense understanding of something without breaks or gaps, has many other significant ramifications. In fact, continuity is one of the most important concepts in mathematics. One might ask then why non-mathematicians should care. The reason is that, on one hand, continuity is a pillar of calculus - another being the idea of a limit - which is essential for the study of engineering and the sciences, while on the other, it has far-reaching consequences in a variety of areas seemingly unconnected with mathematics. Moreover, the absence of continuity can sometimes spell disaster.

To get an idea of what continuity is about, let's consider an example. Suppose you acquired a taste for something, say banana cream pie, long ago, as I did in New York City. Now you crave it but live far away from the city, in a place where no baker seems to know how to make a good banana cream pie. You therefore decide to bake it yourself and so get the recipe. You buy all the ingredients and are ready to make the pie. Now you remember the taste from earlier days, and let's say you are very fussy, i.e., you can tolerate only very small deviations from the taste you remember. If the taste of the pie you make doesn't fall in that narrow band, you are prepared to throw it in the garbage and start afresh. In order to keep the discussion simple, let's pretend that all the ingredients except the key one, say sugar, have been measured out exactly - say they are available in packages. So the only one you have to measure out is sugar. Suppose that the recipe calls for two cups of sugar. Since you can never measure exactly two cups, you may end up putting in slightly more or slightly less than two cups. What about the taste? You **know** that the taste won't be exactly the way you remember it, but will it fall in the band you have chosen? You will of course be happy if it does. More generally, the question would be: for every choice of a narrow band around the remembered taste, is there a range of errors you can make in measuring out the sugar such that the taste falls in the band chosen? In other words, is there an interval containing 2, like say, $(1.9, 2.1)$, such that if the amount of sugar you put in, in units of cups, falls in that interval, then the taste of the pie lies in the band chosen? (By the interval $(1.9, 2.1)$, we mean the set of all numbers between 1.9 and 2.1, excluding 1.9 and 2.1. By contrast, the set of all numbers between 1.9 and 2.1 but including both 1.9 and 2.1 is denoted by $[1.9, 2.1]$.) If that happens, we say that the taste of the pie depends continuously on the amount of sugar put in. More precisely, we say that the taste, as a function of the amount of sugar put in, is continuous at 2. Recall that the choice of 2 was arbitrary. If the same situation holds no matter what we choose in place of 2, we say that the taste of the pie depends continuously on the amount of sugar.

For an example with a scientific flavor, consider the motion of a projectile, such as a rocket launched by NASA. Its motion is governed by the laws of classical mechanics, which are expressed in the form of what are called differential equations (DEs). Attached to the DEs are initial conditions, i.e., the initial position and velocity - the speed and direction - of the rocket. Using these, one can solve the problem, predicting the position and velocity of the rocket at a later time. Now suppose that we change the initial conditions slightly - this can happen because things cannot be measured perfectly. Will the position and velocity at a later time also change slightly, or will they change substantially? It is important to answer this question because otherwise we wouldn't know the position and velocity of the rocket at a later time accurately. Fortunately it can be shown that, under certain conditions, the later position and velocity change only slightly if the change in the initial conditions is small. Mathematicians express this by saying that the solution of the DEs depends continuously on the initial conditions.

In order to understand continuity better, we need a few technical details. First we need to understand the notion of a function, since it is functions that are or aren't continuous. A function is like a computer program. A typical program takes an input and, after doing something to it, yields an output. Similarly, a function is a rule which, when applied to a given number, yields another, possibly the same, number, subject to the condition that whenever the rule is applied to a number, the result is a number and only one number. Thus, for example, the rule "take the positive or negative square root" is not a function because when applied to 4, it yields -2 as well as 2. Notice that this particular rule cannot be applied to negative numbers. The set of all numbers to which a rule can be applied is called its domain, and the set of all numbers obtained as a result of the application of the rule its range. Thus a function f has a domain and a range associated with it. Let x be a member - any member - of the domain of a function f. By the equation $y = f(x)$ we mean that the number y is obtained by applying the rule f to x; we say that y is the value of the function f at x. For an example of a function, let $f(x) = x^2$. Then the domain of f is the set of all real numbers, denoted by $\mathbb{R}$, while its range is the set of all non-negative numbers. (A proper description of $\mathbb{R}$ is beyond the scope of this essay, but for our purposes, we can think of it as the set of all finite numbers - positive, negative, and zero.) The graph of f is a parabola that passes through the origin - meaning that $y = 0$ when $x = 0$ - and has the y - axis as its axis of symmetry.

As x varies over the domain of a function f, y varies over its range. Observe that we can vary x as we please within the domain of f. However, once we choose a specific value of x, we have no control over the resulting value of y, for that is determined by the function. For this reason, x is called the independent variable and y the dependent variable. Still, we seek some control over the value of y. To see why, let's again consider computer programs. Suppose we are interested in a specific output and know an input that would lead to that output. What if we changed the input slightly? Would the output also change slightly, or would it change drastically? It is out of attempts to answer questions like these that the idea of continuity arises.

Before defining continuity, let's consider another example. Recall that, in the real world, there is no such thing as absolute precision. Suppose that in order to make one pill of a certain drug, a pharmaceutical company needs a micrograms of some chemical. (Usually several ingredients are needed to make a drug, but suppose for the sake of simplicity that all of them except this one chemical are somehow held constant. In other words, we will pretend that, for our purposes, this particular chemical is the only ingredient needed.) Is it possible to measure out that exact amount? Obviously the answer is no. However, modern technology enables the company to make the difference between a and the amount measured out extremely small, i.e., very, very close to 0 but not exactly 0. A natural question arises: how does this minute error affect the nature of the drug? We **know** that if the amount of the ingredient is exactly a, the pill will have the exact, desired effect. But what if the amount differs ever so slightly from a? That's where continuity comes in. Suppose that quality control experts have constructed a function f which measures the deviation of the quality of the drug from what it would be if $x = a$; thus $f(x) = 0$ if $x = a$, indicating that there would be no change in the quality of the drug if x were exactly equal to a. For f to be useful, it is not enough if a is in its domain; there must also be an interval containing a such that every point in it is in the domain of f. So let us assume that to be the case. Now as x varies around a, $f(x)$ varies around 0, with 0 indicating no change in the quality and, say, 1 indicating extreme variation. Suppose that engineers have built a machine which rejects a pill if, for that pill, $f(x)$ exceeds some preset tolerance level chosen by pharmacologists. We would say that f is continuous at a if for every tolerance level chosen by the pharmacologists, we can find an interval containing a such that whenever x lies in that interval - observe that x can be either smaller than a or larger than a; hence the need for an interval - $f(x)$ lies between 0 and the tolerance level, i.e., the machine would accept the pill. Thus, if f is continuous at a, the pharmaceutical company need only make sure that the amount x of the chemical measured out to make a pill lies within an interval containing a, which is determined by the preset tolerance level.

The examples we have considered so far show that continuity is a desirable property but don't tell us what happens if a function f is **not** continuous at a point a. This happens if either a is not in the domain of f or if it is in the domain of f but a small change in x from a causes a big change in $f(x)$ from $f(a)$. We say a function is discontinuous at a point if it is not continuous there. Now let us consider another example. We human beings are fallible: most of us, most of the time, want to stay on the straight and narrow path - the proverbial razor's edge - but end up straying from it once in a while. Most of us would consider a society civilized if, in it, punishment for deviations from the straight and narrow path is **commensurate** with the deviation; thus, if an infraction is small, so would be the punishment for it. Suppose that, like in the example of the pill, ethicists have come up with a function f which measures the severity of punishment, and suppose that its values go from 0 to 1, 0 indicating no punishment at all and 1 indicating extreme punishment; thus if x is a measure of the deviation from the straight and narrow path, we would have $f(0) = 0$; we would regard a society as civilized if whenever x differs slightly from 0, $f(x)$ also differs slightly from 0, i.e., if f is continuous at 0. Now, here's an interesting example that actually took place a while ago. A Greyhound bus pulled in to the bus station in San Diego, California. People were getting on and off, but some passengers headed for other destinations continued to sit in their seats. In that hustle and bustle, a passenger boarding the bus bumped into a man who was sitting, a natural but minor infraction. The passenger who was sitting was so enraged that he pulled out a gun and shot the other guy dead, on the spot. Now **that** is discontinuous!

Finally we are ready to define continuity, or more precisely, the continuity of a function f at a point a in its domain. This means that we let x be a specific number a - it doesn't matter which one - in the domain of f. Mathematicians use symbols to define continuity, but we will skirt around that and give a visual and intuitive definition. To make things simple, let's say that the domain of f is the interval $[0, 1]$. We say f is continuous at a if the graph of f doesn't have a break or a gap at a. Notice that this property is local in the sense that it tells us something about the behavior of f at a but says nothing at all about its behavior at another point, say b, in its domain. If f is continuous at every point in its domain, we say simply that f is continuous. Suppose that is the case. Then one can draw its graph without lifting one's pen or pencil. Thus the graph would be an unbroken curve connecting the points $(0, f(0))$ and $(1, f(1))$.

Moving from the definition of continuity to that of a limit, the other pillar of calculus, is easy and straightforward. (Traditionally, it is done the other way around, but since limits are harder to understand intuitively, this is easier.) We simply drop from the definition of continuity the restriction that a be in the domain of f. Observe, however, that then we can no longer speak of how $f(x)$ differs from $f(a)$, for the latter may not exist So instead, we wonder if there is a real number L such that as x varies slightly from a, $f(x)$ also varies only slightly from L. If such a number L exists, we call it the limit of $f(x)$ as x approaches a. With an understanding of these fundamental ideas (continuity and limits) we are well-equipped to embark upon the study of calculus as well as analysis, the branch of mathematics that provides the theoretical underpinnings of calculus.

Finally, continuity, together with a condition called compactness, implies many interesting and useful properties. (Compactness is a property of a set, in our case the domain of a function f. A simple example of a compact set is the interval $[0, 1]$.) For example, if a function f is continuous on $[0, 1]$, then there is a largest value of $f(x)$ as x varies over $[0, 1]$ as well as a smallest value. More generally, this would be true if f is continuous on a compact set E. This is very useful if, for instance, $f(x)$ represents the cost of producing x units of something. Then one might want to find x for which $f(x)$ is a minimum. In another case, we might wish to maximize a quantity such as income or profit. Many interesting and useful things happen in the presence of continuity and compactness. Indeed, that combination is a real cornucopia.

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