MATH 2135, LINEAR ALGEBRA, Winter 2017

Handout 3: Problems on functions

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Recall the definitions of direct image and preimage:

$$\begin{aligned} f(X) &= \{y \mid \text{there exists } x \in X \text{ such that } y = f(x)\} \\ f^{-1}(Y) &= \{x \mid f(x) \in Y\}. \end{aligned}$$

Also recall the definitions of one-to-one and onto functions from Chapter 5.2.

Problem 1. Let A, B be sets and $f : A \to B$ be a function. Prove that:

(a) For all $X \subseteq A$ and $Y \subseteq B$, we have $f(X) \subseteq Y$ iff $X \subseteq f^{-1}(Y)$.

Hint: your proof should start like this: "We prove both directions of the implication. First, assume $f(X) \subseteq Y$. To show $X \subseteq f^{-1}(Y)$, take an arbitrary $x \in X$. By definition of $f(X)$, it follows that $f(x) \in f(X)$. ...Conversely, assume $X \subseteq f^{-1}(Y)$. To show $f(X) \subseteq Y$, take an arbitrary $y \in f(X)$. ...

(b) For all $X \subseteq A$, we have $X \subseteq f^{-1}(f(X))$.

(c) For all $Y \subseteq B$, we have $f(f^{-1}(Y)) \subseteq Y$.

Hint: use part (a) to prove parts (b) and (c).

Problem 2. Let A, B be sets and $f : A \to B$ be a function. Prove that:

(a) f is one-to-one iff for all $X \subseteq A$, we have $X = f^{-1}(f(X))$.

Hint: your proof should start like this: "We prove both directions of the implication. First, assume f is one-to-one, and let $X \subseteq A$ be some arbitrary subset. From the previous problem, we already know that $X \subseteq f^{-1}(f(X))$. We have to show that $f^{-1}(f(X)) \subseteq X$. So let $x \in f^{-1}(f(X))$ be an arbitrary element. ...For the opposite implication, assume that for all $X \subseteq A$, we have $X = f^{-1}(f(X))$. We wish to show that f is one-to-one. Consider, therefore, two elements $x, x' \in A$ such that $f(x) = f(x')$. We have to show that $x = x'$. ...

(b) f is onto iff for all $Y \subseteq B$, we have $f(f^{-1}(Y)) = Y$.

Problem 3. Let V and U be vector spaces over some field K. Prove that a function $f : V \to U$ is linear if and only if for all scalars $a, b \in K$ and all vectors $v, w \in V$,

$$f(av + bw) = af(v) + bf(w).$$

Problem 4. Prove Proposition 5.4: Suppose $v_1, \ldots, v_m$ span a vector space V, and suppose $f : V \to U$ is linear. Then $f(v_1), \ldots, f(v_m)$ span Im f.

Problem 5. Let $V = P_n(t)$, and consider the map $f : V \to V$ such that for every polynomial $p \in P_n(t)$, $f(p) = p'$, where p' is the derivative of p (see Example 5.5, p.168). What is the kernel of f? What is the image of f? What is the rank of f?