MATH 2135, LINEAR ALGEBRA, Winter 2017

Handout 4: Problems

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Problem 1. Calculate the determinants of the following matrices (a) over the field $\mathbb{R}$ of real numbers, (b) over the field $\mathbb{Z}_5$.

$$A = \begin{pmatrix} 1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 4 & 3 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 4 & 2 & 1 & 2 \\ 0 & 0 & 0 & 3 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}$$

Problem 2. Consider the vector space $\mathbf{P}_2(t)$ of polynomials of degree at most 2. Consider the linear function $f : \mathbf{P}_2(t) \to \mathbf{P}_2(t)$ defined by: $f(p(t)) = p(t+2)$. For example,

$$f(t^2 + 2t + 1) = (t+2)^2 + 2(t+2) + 1 = t^2 + 4t + 4 + 2t + 4 + 1 = t^2 + 6t + 9.$$

Consider the basis $S = \{1, t, t^2\}$ of $\mathbf{P}_2(t)$. Find the matrix representation of f with respect to the basis S. What is the determinant of f?

Problem 3. Consider the formula for the determinant of an $n \times n$-matrix A:

$$\det A = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}.$$

Using this formula, prove that if a matrix A has two identical rows, then $\det A = 0$. (Hint: to see what is going on, first try this for $n = 3$).

Problem 4. Find an orthonormal basis of $\mathbb{R}^3$ containing the vector $\frac{1}{3}(1, 2, 2)$ as the first basis vector.

Problem 5. Consider the subspace of $\mathbb{R}^4$ spanned by $(1, 1, 0, 0)$, $(1, 0, 1, 0)$, and $(1, 0, 0, 1)$. Use the Gram-Schmidt method to find an orthogonal basis of this subspace.

Problem 6. On $\mathbb{R}^3$, consider the inner product defined by $\langle v, w \rangle = v^T A w$, where

$$A = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 3 \end{pmatrix}$$

Use the Gram-Schmidt method to find a basis of $\mathbb{R}^3$ that is orthonormal with respect to this inner product.

Problem 7. Consider the vector space $V = C[0, 1]$ of continuous, real-valued functions defined on the unit interval $[0, 1] = \{x \in \mathbb{R} \mid 0 \leqslant x \leqslant 1\}$. Consider the inner product on V that is defined by

$$\langle f, g \rangle = \int_0^1 f(x) g(x)\, dx.$$

Let $W \subseteq V$ be the subspace spanned by the following three functions: $f_0(x) = 1$, $f_1(x) = x$, and $f_2(x) = x^2$.

(a) Calculate the inner products $\langle f_i, f_j \rangle$ for all $i, j \in \{0, 1, 2\}$.

(b) Using the Gram-Schmidt method starting from $\{f_0, f_1, f_2\}$, find an orthonormal basis for W.

(c) **Approximation.** Consider the function g on $[0, 1]$ defined by $g(x) = x^3$. Find the best quadratic approximation of g, i.e., find the quadratic function $h \in W$ such that

$$\int_0^1 (h(x) - g(x))^2\, dx$$

is as small as possible. Hint: this is equivalent to requiring that $\|h - g\|$ is as small as possible, i.e., h is the orthogonal projection of g onto the subspace W.

Problem 8. (a) Find the characteristic polynomial, eigenvalues, and eigenvectors of the following matrices. (b) Determine the algebraic multiplicity and geometric multiplicity of each eigenvalue. (c) Which of the matrices can be diagonalized? Diagonalize them.

$$A = \begin{pmatrix} 5 & -6 \\ 3 & -4 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & 1 & -1 \\ -1 & -2 & 1 \\ 0 & -4 & 3 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 3 & -1 \\ 0 & 0 & 2 \end{pmatrix}$$

Problem 9. Find a unitary matrix U and a diagonal matrix D such that $U^* A U = D$.

$$A = \begin{pmatrix} 3 & -2 & 0 \\ -2 & -1 & 1 \\ 0 & 1 & 3 \end{pmatrix}$$

Problem 10. A matrix B is called a *square root* of a matrix A if $B^2 = A$. Find four different square roots of A. (Hint: work relative to a basis in which A is diagonal, then convert the answer to the original basis).

$$A = \begin{pmatrix} -2 & -6 \\ 3 & 7 \end{pmatrix}$$

(Remember that it is easy to double-check your answer).

Problem 11. Find the matrix A whose eigenvalues are $\lambda_1 = 1$, $\lambda_2 = -1$, and $\lambda_3 = 0$, with respective eigenvectors

$$v_1 = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \qquad v_3 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

(Remember that it is easy to double-check your answer).

Problem 12. Consider the sequence of numbers $a_0, a_1, a_2, a_3, \ldots$ obtained by the following rule: $a_0 = 1$, $a_1 = 1$, and for all $n \geqslant 1$, $a_{n+1} = a_n + 2a_{n-1}$. Therefore, the sequence starts with 1 and 1, and the next number is calculated as the sum of the current number plus twice the previous number. The first few elements of this sequence are:

$$1, 1, 3, 5, 11, 21, 43, \ldots$$

The goal of this exercise is to find a direct formula for the nth element of this sequence.

Let $v_n = \begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix}$ denote the vector consisting of the nth and the $n+1$st elements of the sequence. Then we have the following relationship:

$$v_0 = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

and

$$v_n = \begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix} = \begin{pmatrix} a_n \\ a_n + 2a_{n-1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a_{n-1} \\ a_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} v_{n-1},$$

for all $n \geqslant 1$. We therefore obtain the following formula for v_n:

$$v_n = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}^n \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Note that this formula involves raising the matrix $A = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$ to the nth power.

(a) Find the eigenvalues and eigenvectors of A.

(b) Find an invertible P and a diagonal matrix D such that $D = P^{-1}AP$.

(c) Prove that $A^n = PD^nP^{-1}$, for all n.

(d) Using (c) and (b), give an explicit formula for A^n.

(e) Give an explicit formula for v_n.

(f) Give an explicit formula for a_n.

(g) Check your formula by using it to compute the first few elements of the sequence.

The following problems are additional proof drills.

Problem 13. Let $f : V \to W$ be a linear function, and assume f is one-to-one. Let $v_1, \ldots, v_n \in V$ be linearly independent. Prove that $f(v_1), \ldots, f(v_n)$ are linearly independent.

Problem 14. Let $f : V \to W$ be a linear function, and assume $v_1, \ldots, v_n \in V$ are points such that $f(v_1), \ldots, f(v_n)$ are linearly independent. Prove that $v_1, \ldots, v_n$ are linearly independent.

Problem 15. Let $f : V \to W$ be a linear function. Prove that $\ker f$ is a subspace of V. Also prove that $\operatorname{Im} f$ is a subspace of W.

Problem 16. Let $f : V \to W$ be a linear function, and let $U \subseteq V$ be a subspace of V. Recall the definition of direct image:

$$f(U) = \{w \in W \mid \text{there exists } u \in U \text{ with } f(u) = w\}.$$

Prove that $f(U)$ is a subspace of W.

Problem 17. Let $f : V \to W$ be a linear function, let $v_1, \ldots, v_m \in V$ be a basis of the kernel of f, and let $w_1, \ldots, w_p \in W$ be a basis of the image of f. Let $u_1, \ldots, u_p \in V$ be vectors such that $f(u_1) = w_1, \ldots, f(u_p) = w_p$. Prove that $\{v_1, \ldots, v_m, u_1, \ldots, u_p\}$ is a basis of V.