MAT 3361, INTRODUCTION TO MATHEMATICAL LOGIC, Fall 2004

Answers to the Midterm

Problem 1. Prove the following in natural deduction, using your choice of Fitch or Prawitz style:

(a) $(A \to B) \to C \quad \vdash \quad A \vee C.$

Answer:

1	$(A \longrightarrow B) \longrightarrow C$	
2	$\mid\ \neg(A \vee C)$	
3	$\mid\ \mid\ A$	
4	$\mid\ \mid\ A \vee C$	$\vee$I, 3
5	$\mid\ \mid\ \bot$	$\neg$E, 2, 4
6	$\mid\ \mid\ B$	$\bot$E, 5
7	$\mid\ A \longrightarrow B$	$\Rightarrow$I, 3–6
8	$\mid\ C$	$\Rightarrow$E, 1, 7
9	$\mid\ A \vee C$	$\vee$I, 8
10	$\mid\ \bot$	$\neg$E, 2, 9
11	$\neg\neg(A \vee C)$	$\neg$I, 2–10
12	$A \vee C$	$\neg\neg$E, 11

(b) $A \vee B \quad \vdash \quad (\neg B) \to (C \to A).$

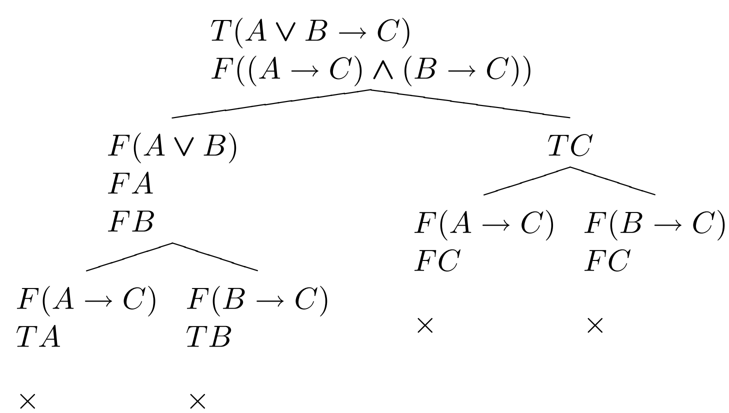
Answer:

1	$A \vee B$	
2	$\mid\ \neg B$	
3	$\mid\ \mid\ C$	
4	$\mid\ \mid\ \mid\ A$	
5	$\mid\ \mid\ \mid\ A$	R, 4
6	$\mid\ \mid\ \mid\ B$	
7	$\mid\ \mid\ \mid\ \bot$	$\neg$E, 2, 6
8	$\mid\ \mid\ \mid\ A$	$\bot$E, 7
9	$\mid\ \mid\ A$	$\vee$E, 1, 4–5, 6–8
10	$\mid\ C \longrightarrow A$	$\Rightarrow$I, 3–9
11	$\neg B \longrightarrow (C \longrightarrow A)$	$\Rightarrow$I, 2–10

Problem 2. Prove the following proposition using analytic tableaux:

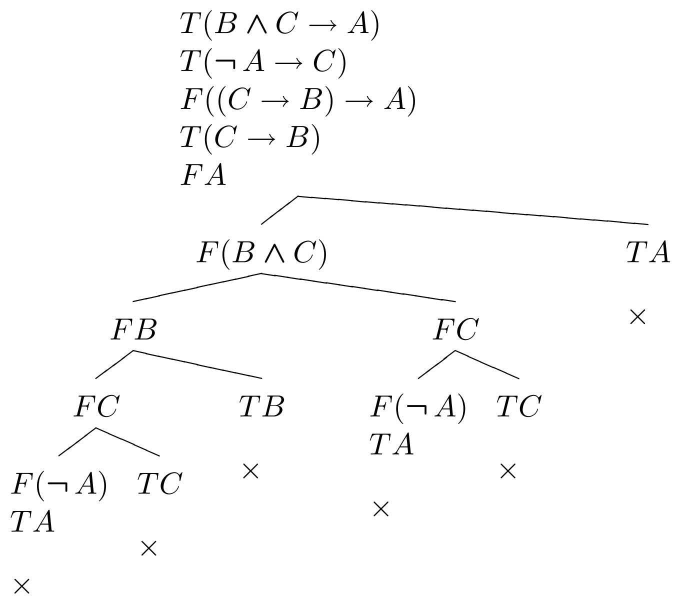
(a) $(A \vee B) \to C \quad \vdash \quad (A \to C) \wedge (B \to C).$

Answer:



(b) $B \wedge C \to A, \quad (\neg A) \to C \quad \vdash \quad (C \to B) \to A.$

Answer:



Problem 3. Let us write $v(\varphi)$ for the number of occurrences of propositional variables in a proposition φ. Let us write $c(\varphi)$ for the number of occurrences of connectives in φ.

Examples:

$v(((p_2 \wedge p_5) \to (\bot \wedge p_2))) = 3$, because there are 3 variable occurrences p_2, p_5, p_2.

$c(((p_2 \wedge p_5) \to (\bot \wedge p_2))) = 4$, because there are 4 occurrences of connectives $\wedge, \to, \bot, \wedge$.

(a) Give recursive definitions of $v(\varphi)$ and $c(\varphi)$.

Answer:

$$\begin{aligned}
v(p_i) &= 1 \\
v(\bot) &= 0 \\
v((\varphi \,\square\, \psi)) &= v(\varphi) + v(\psi) \\
v((\neg \varphi)) &= v(\varphi)
\end{aligned}$$

$$\begin{aligned}
c(p_i) &= 0 \\
c(\bot) &= 1 \\
c((\varphi \,\square\, \psi)) &= c(\varphi) + c(\psi) + 1 \\
c((\neg \varphi)) &= c(\varphi) + 1
\end{aligned}$$

(b) Prove: for all φ, $v(\varphi) \leqslant c(\varphi) + 1$.

Answer: Base case: $v(p_i) = 1 \leqslant 0 + 1 = c(p_i) + 1$, and $v(\bot) = 0 \leqslant 1 + 1 = c(\bot) + 1$.

Induction step: let $\varphi = (\psi \,\square\, \theta)$. By induction hypothesis, $v(\psi) \leqslant c(\psi) + 1$ and $v(\theta) \leqslant c(\theta) + 1$. We want to show $v(\varphi) \leqslant c(\varphi) + 1$. But:

$$\begin{aligned}
v(\varphi) &= v(\psi \,\square\, \theta) \\
&= v(\psi) + v(\theta) \\
&\leqslant c(\psi) + 1 + c(\theta) + 1 \text{ by ind.hyp.} \\
&= c(\psi \,\square\, \theta) + 1 \\
&= c(\varphi) + 1
\end{aligned}$$

Second induction step: let $\varphi = (\neg \psi)$. By induction hypothesis, $v(\psi) \leqslant c(\psi) + 1$. We want to show $v(\varphi) \leqslant c(\varphi) + 1$. But:

$$\begin{aligned}
v(\varphi) &= v(\neg \psi) \\
&= v(\psi) \\
&\leqslant c(\psi) + 1 \text{ by ind.hyp.} \\
&= c(\neg \psi) \\
&= c(\varphi) \\
&\leqslant c(\varphi) + 1
\end{aligned}$$

Problem 4. Suppose $\vdash A \to B$ and $\nvdash B$. Prove: the set $\{\neg A, \neg B\}$ is consistent.

Answer: There are many possible proofs. Here are two examples. Note that proof 1 uses soundness and completeness, whereas proof 2 does not.

Proof 1: Using soundness, we have $\models A \to B$ and $\not\models B$. Because $\not\models B$, it follows that there exists some valuation $[\![-]\!]_0$ such that $[\![B]\!]_0 = 0$. But by $\models A \to B$, it follows that for all valuations, $[\![-]\!]$, $[\![B]\!] = 0$ implies $[\![A]\!] = 0$. So in particular, $[\![A]\!]_0 = 0$. So then, $[\![\neg B]\!]_0 = 1$ and $[\![\neg A]\!]_0 = 1$, so we have found a valuation which satisfies $\{\neg A, \neg B\}$. It follows, by completeness, that $\{\neg A, \neg B\}$ is consistent. $\square$

Proof 2: Suppose $\vdash A \to B$ and $\nvdash B$. Also, suppose, for the sake of contradication, that $\{\neg A, \neg B\}$ is not consistent. By definition, this means that $\neg A, \neg B \vdash \bot$. We then have the following natural deduction proofs:

$$\dfrac{\vdots}{A \to B} \qquad\qquad \dfrac{\dfrac{\neg A}{\vdots} \quad \dfrac{\neg B}{\vdots}}{\bot}$$

We can therefore construct the following natural deduction proof:

$$\dfrac{\dfrac{[\neg B]_2 \quad \dfrac{\dfrac{\vdots}{A \to B} \quad \dfrac{\dfrac{\dfrac{[\neg A]_1}{\vdots} \quad \dfrac{[\neg B]_2}{\vdots}}{\bot}}{A}\,(RAA_1)}{B}\,(\to E)}{\bot}\,(\neg E)}{B}\,(RAA_2)$$

We therefore have $\vdash B$, contradicting our assumption. $\square$

Problem 5. Let Γ be a set of propositions such that for all propositional variables p_n, either $\Gamma \models p_n$ or $\Gamma \models \neg p_n$. Prove by induction: for all propositions φ, either $\Gamma \models \varphi$ or $\Gamma \models \neg \varphi$.

Note: *To keep the problem short, do only the cases $\{atoms, \neg, \wedge\}$. Make sure you state the induction hypothesis clearly in each case.*

Answer: Base case: if $\varphi = p_n$, then the claim is true by assumption.

Induction step ($\neg$): suppose $\varphi = \neg \psi$, and suppose that $\Gamma \models \psi$ or $\Gamma \models \neg \psi$ (induction hypothesis). We want to show $\Gamma \models \varphi$ or $\Gamma \models \neg \varphi$. Case 1: if $\Gamma \models \psi$, then $\Gamma \models \neg\neg\psi$ (because for any valuation, $[\![\neg\neg\psi]\!] = [\![\psi]\!]$. Therefore $\Gamma \models \neg\varphi$. Case 2: if $\Gamma \models \neg\psi$, then $\Gamma \models \varphi$, because $\varphi = \neg\psi$.

Induction step ($\wedge$): suppose $\varphi = \psi \wedge \rho$, and suppose that ($\Gamma \models \psi$ or $\Gamma \models \neg\psi$) and ($\Gamma \models \rho$ or $\Gamma \models \neg\rho$) (induction hypothesis). We want to show $\Gamma \models \varphi$ or $\Gamma \models \neg\varphi$. Case 1: if $\Gamma \models \psi$ and $\Gamma \models \rho$, then $\Gamma \models \psi \wedge \rho$, hence $\Gamma \models \varphi$. Case 2: if $\Gamma \models \neg\psi$, then $\Gamma \models \neg(\psi \wedge \rho)$, because $\neg\psi$ logically implies $\neg(\psi \wedge \rho)$. Therefore, $\Gamma \models \neg\varphi$. Case 3: if $\Gamma \models \neg\rho$, then $\Gamma \models \neg(\psi \wedge \rho)$, because $\neg\rho$ logically implies $\neg(\psi \wedge \rho)$. Therefore, $\Gamma \models \neg\varphi$. Since at least one of the three cases must hold, we are done. $\square$

Problem 6. In this problem, we consider a version of Hintikka sets for unsigned propositions. For simplicity, we consider a propositional logic using only the connectives $\{\neg, \wedge\}$. Thus, we do not consider the connectives $\{\vee, \to, \leftrightarrow, \bot\}$

Let S be a set of unsigned propositions. S is called a *Hintikka set* if it satisfies:

1. for no propositional symbol p, both $p \in S$ and $(\neg p) \in S$,
2. if $(\varphi \wedge \psi) \in S$, then $\varphi \in S$ and $\psi \in S$,
3. if $(\neg(\varphi \wedge \psi)) \in S$, then $(\neg\varphi) \in S$ or $(\neg\psi) \in S$,

4. if $(\neg(\neg\varphi)) \in S$, then $\varphi \in S$.

(a) Prove: Every Hintikka set is satisfiable.

Hint: first, define a suitable valuation $[\![-]\!]$. Then, prove the following statement by induction: for all propositions φ, ($\varphi \in S \Rightarrow [\![\varphi]\!] = 1$) and ($(\neg\varphi) \in S \Rightarrow [\![\varphi]\!] = 0$).

Answer: Let S be a Hintikka set. Define a valuation $[\![-]\!]$ by:

$$[\![p_i]\!] = \begin{cases} 1 & \text{if } p_i \in S, \\ 0 & \text{if } p_i \notin S, \end{cases}$$

and extend $[\![-]\!]$ to composite formulas in the unique way:

$$\begin{array}{lcl} [\![\neg\psi]\!] & = & 1 - [\![\psi]\!], \\ [\![\psi \wedge \rho]\!] & = & \min\{[\![\psi]\!], [\![\rho]\!]\}. \end{array}$$

We claim that for all propositions φ, $\varphi \in S \Rightarrow [\![\varphi]\!] = 1$ and $(\neg\varphi) \in S \Rightarrow [\![\varphi]\!] = 0$. We prove this by induction on φ.

Base case: if $\varphi = p_i$ is atomic, then $p_i \in S \Rightarrow [\![p_i]\!] = 1$ by definition. Also, assume $\neg p_i \in S$, then $p_i \notin S$ by clause (1) in the definition of a Hintikka set, so $[\![p_i]\!] = 0$ by definition.

Induction step ($\wedge$): Assume $\varphi = \psi \wedge \rho$, and assume the induction hypothesis holds for ψ and ρ. If $\varphi \in S$, then by clause (2) in the definition of a Hintikka set, $\psi \in S$ and $\rho \in S$. By I.H., $[\![\psi]\!] = 1$ and $[\![\rho]\!] = 1$, hence $[\![\varphi]\!] = [\![\psi \wedge \rho]\!]{=}1$. If $\neg\varphi \in S$, then by clause (3) in the definition of a Hintikka set, $\neg\psi \in S$ or $\neg\rho \in S$. Without loss of generality, assume $\neg\psi \in S$. Then by I.H., $[\![\psi]\!] = 0$, hence $[\![\varphi]\!] = [\![\psi \wedge \rho]\!] = 0$.

Induction step ($\neg$): Assume $\varphi = \neg\psi$, and assume the induction hypothesis holds for ψ. If $\varphi \in S$, then $\neg\psi \in S$, hence $[\![\psi]\!]{=}0$ by I.H., hence $[\![\varphi]\!]{=}1$. If $\neg\varphi \in S$, then $\neg\neg\psi \in S$, and by clause (4) in the definition of a Hintikka set, $\psi \in S$, hence by I.H., $[\![\psi]\!] = 1$, hence $[\![\neg\varphi]\!] = 1$.

It follows that $[\![-]\!]$ satisfies all $\varphi \in S$, so S is satisfiable. $\square$

(b) [Extra credit] Prove: Every maximally consistent set is a Hintikka set.

Answer: Let S be a maximally consistent set of formulas. From a theorem in class, we know that every maximally consistent set satisfies $\varphi \in S \iff \neg\varphi \notin S$, for all φ. We prove that S is a Hintikka set:

(1) Since S is consistent, we cannot have $p \in S$ and $\neg p \in S$, or else $S \vdash \perp$.

(2) Assume $\varphi \wedge \psi \in S$. Suppose $\varphi \notin S$. Then $\neg\varphi \in S$ by maximality. But $\varphi \wedge \psi, \neg\varphi \vdash \perp$, so S is inconsistent, a contradiction. Therefore $\varphi \in S$.

(3) Assume $\neg(\varphi \wedge \psi) \in S$. Suppose $\neg\varphi \notin S$ and $\neg\psi \notin S$. By maximal consistency, $\varphi \in S$ and $\psi \in S$. But $\neg(\varphi \wedge \psi), \varphi, \psi \vdash \perp$, so S is inconsistent, a contradiction. Therefore $\neg\varphi \in S$ or $\neg\psi \in S$.

(4) Assume $\neg\neg\varphi \in S$. By maximal consistency, $\neg\varphi \notin S$, and therefore $\varphi \in S$. We have shown that S satisfies all 4 conditions of a Hintikka set.