

Lecture 7

Topological model of propositional intuitionistic logic

We can check the validity of propositional judgements in classical logic using Venn diagrams: assign a subset of $\mathbb{R}^2$ to each propositional variable, then interpret "and" as intersection, "or" as union, and "not" as complement. A judgement $A_1, \dots, A_n \vdash B$ is valid iff for all such interpretations, $A_1 \cap \dots \cap A_n \subseteq B$.

There is a similar interpretation for intuitionistic propositional logic too. This time, we assign an open subset of $\mathbb{R}^2$ to each propositional variable. We still interpret "and" as intersection and "or" as union. The interpretations of negation and implication change:

- $\neg A = (A^c)^\circ$.
- $A \rightarrow B = (A^c \cup B)^\circ$.

Here, A^c denotes the complement of A , i.e., $\mathbb{R}^2 \setminus A$, and X° denotes the interior of a set X , i.e., the largest open set contained in X .

Theorem: A judgement $A_1, \dots, A_n \vdash B$ is valid iff for all such interpretations, $A_1 \cap \dots \cap A_n \subseteq B$.

Topological model of propositional intuitionistic logic.

To each proposition, assign an open subset of $\mathbb{R}^2$.



Connectives:

$A \wedge B$: Intersection $A \cap B$

$A \vee B$: Union $A \cup B$

Negation: $\neg A$: $(\mathbb{R}^2 - A)^{\circ}$ = The interior of the complement!

Implication:

$$A \rightarrow B : ((\mathbb{R}^2 - A) \cup B)^{\circ}$$

Interior.



Judgement $A_1, \dots, A_n \vdash B$

"holds" in the interpretation if

$$A_1 \cap \dots \cap A_n \subseteq B$$

We can use this to give counterexamples to propositional intuitionistic judgements, i.e., to show that something is not valid.

Example: Excluded middle.

Let $A = \{v \in \mathbb{R}^2 \mid |v| < 1\}$.

Then $\neg A = (A^c)^{\circ} = \{v \in \mathbb{R}^2 \mid |v| > 1\}$.

Then $A \vee \neg A = A \cup \neg A = \{v \in \mathbb{R}^2 \mid |v| \neq 1\}$.

Since $\mathbb{R}^2 \not\subseteq A \vee \neg A$, the judgement $\vdash A \vee \neg A$ is not valid.

Counterexamples

Excluded middle $\not\vdash A \vee \neg A$

$A = \{v \mid |v| < 1\}$

$\neg A = \{v \mid |v| > 1\}$

$A \vee \neg A = \{v \mid |v| \neq 1\}$

$A_1, A_2, \dots, A_n \vdash B$

$A_1 \cap \dots \cap A_n \subseteq B$

DNE

$\neg A \not\vdash A$

$A = \{v \mid |v| < 1\}$

$\neg A = (\mathbb{R}^2 - A)^{\circ} = \{v \mid |v| > 1\}$

$\neg \neg A = \mathbb{R}^2$

$\mathbb{R}^2 \not\subseteq A$

$\Rightarrow \neg A \not\vdash A$

Example: Double negation elimination.

Let $A = \{v \in \mathbb{R}^2 \mid v \neq 0\}$.

Then $\neg A = (A^c)^0 = \emptyset$.

Then $\neg\neg A = ((\neg A)^c)^0 = \mathbb{R}^2$.

Since $\neg\neg A \not\subseteq A$, the judgement $\neg\neg A \vdash A$ is not valid.

Example: De Morgan's law $\neg(A \wedge B) \vdash \neg A \vee \neg B$ fails intuitionistically.

Let $A = \{(x, y) \in \mathbb{R}^2 \mid x < 0\}$.

Let $B = \{(x, y) \in \mathbb{R}^2 \mid x > 0\}$.

Then $A \wedge B = \emptyset$.

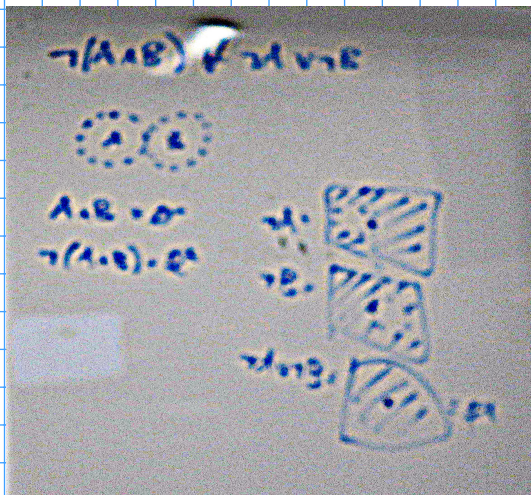
Then $\neg(A \wedge B) = \mathbb{R}^2$.

Also, $\neg A = \{(x, y) \in \mathbb{R}^2 \mid x > 0\}$.

Also, $\neg B = \{(x, y) \in \mathbb{R}^2 \mid x < 0\}$.

Therefore, $\neg A \vee \neg B = \{(x, y) \in \mathbb{R}^2 \mid x \neq 0\}$.

Since $\neg(A \wedge B) \not\subseteq \neg A \vee \neg B$, the judgement $\neg(A \wedge B) \vdash \neg A \vee \neg B$ is not valid.



Example: Peirce's Law $((A \rightarrow B) \rightarrow A) \vdash A$ is valid classically, but not intuitionistically.

Let $A = \mathbb{R}^2 \setminus \{0\}$ and $B = \emptyset$.

Then $A \rightarrow B = \emptyset$ and $(A \rightarrow B) \rightarrow A = \mathbb{R}^2$.

Since $(A \rightarrow B) \rightarrow A \not\subseteq A$, the judgement $((A \rightarrow B) \rightarrow A) \vdash A$ is not valid.

Peirce's Law

$((A \rightarrow B) \rightarrow A) \vdash A$

is classically valid.

Intuitionistically, not valid.

$A = \text{[diagram of a square with a circle inside, shaded outside the circle]}$ $B = \emptyset$

$A \rightarrow B = \emptyset$

$(A \rightarrow B) \rightarrow A = \emptyset \rightarrow A = \mathbb{R}^2$

$A_1, A_2, \dots, A_n \vdash B$

$A_1, \dots, A_n \subseteq B$

$\text{[diagram of a square with a circle inside, shaded outside the circle]} = \{v \mid \|v\| > 1\}$

DNE

$\neg A \not\vdash A$

$A = \text{[diagram of a square with a circle inside, shaded outside the circle]}$

$\neg A = \text{[diagram of a square with a circle inside, unshaded]} = \emptyset$

$\neg \neg A = \mathbb{R}^2$

$\mathbb{R}^2 \not\subseteq A$

$\Rightarrow \neg \neg A \not\vdash A$

Quantifiers: The Brouwer-Heyting-Kolmogorov interpretation of "for all" and "there exists":

Brouwer-Heyting-Kolmogorov.

- A proof of $\forall (x:A), P(x)$
is a function that inputs $x:A$ and outputs a proof of $P(x)$.
- A proof of $\exists (x:A), P(x)$
is a pair of an $x:A$ and a proof of $P(x)$.