MAT 5361, TOPICS IN QUANTUM COMPUTATION, WINTER 2004
Homework 1

Problem 1. Recall the law of measurement of pure quantum states (we give it here for a 2-qubit state, with the left qubit being measured):

$$\begin{tikzcd}[column sep=small]
 & \alpha\,|\mathbf{00}\rangle + \beta\,|\mathbf{01}\rangle + \gamma\,|\mathbf{10}\rangle + \delta\,|\mathbf{11}\rangle \arrow[dl, "p_0=|\alpha|^2+|\beta|^2"', "\mathbf{0}"] \arrow[dr, "\mathbf{1}"', "p_1=|\gamma|^2+|\delta|^2"] & \\
\alpha\,|\mathbf{00}\rangle + \beta\,|\mathbf{01}\rangle & & \gamma\,|\mathbf{10}\rangle + \delta\,|\mathbf{11}\rangle
\end{tikzcd}$$

Here, we assume $|\alpha|^2+|\beta|^2+|\gamma|^2+|\delta|^2=1$, p_0 is the probability of observing $\mathbf{0}$, and p_1 is the probability of observing $\mathbf{1}$. Using this, together with the definition of mixed states, density matrices, and the normalization convention for density matrices, derive the correctness of the density matrix formulation for measurement:

$$\begin{tikzcd}[column sep=small]
 & \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \arrow[dl, "p_0=\mathrm{tr}\,A"', "\mathbf{0}"] \arrow[dr, "\mathbf{1}"', "p_1=\mathrm{tr}\,B"] & \\
\left(\begin{array}{c|c} A & 0 \\ \hline 0 & 0 \end{array}\right) & & \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & D \end{array}\right)
\end{tikzcd}$$

Problem 2. Recall the result on the universality of a the H and CONTROLLED-V gates, as quoted from R. Cleve, "An introduction to quantum complexity theory":

Theorem. *Let B be any two-qubit gate and $\epsilon > 0$. Then there exists a quantum circuit of size $O(\log^d(1/\epsilon))$ (where d is a constant) consisting of only H and* CONTROLLED-V *gates which computes a unitary operation B' that approximates B in the following sense. There exists a unit complex number λ (i.e. with $|\lambda|=1$) such that $\|B-\lambda B'\|_2 \leqslant \epsilon$.*

Here, the "O"-notation ("big-oh" notation) is a notation from computer science. It means that there exists a constant c such that the size of the quantum circuit is $\leqslant c\log^d(1/\epsilon)$. The purpose of this problem is to figure out whether this is "little" or "a lot".

(a) The *operator norm* of a matrix A is defined as $\|A\| = \sup\{|Av|;|v| \leqslant 1\}$. Prove that the above theorem holds equally if the 2-norm $\|-\|_2$ is replaced by the operator norm.

(b) Prove $\|AB\| \leqslant \|A\|\|B\|$ and $\|A\otimes B\| = \|A\|\|B\|$.

(c) Prove that errors in quantum circuits are propagated *linearly*, i.e., for a quantum circuit which consists of n gates, if each gate is approximated within ϵ, then the whole circuit is approximated within $n\epsilon$. This is in contrast to classical analog circuits and floating point arithmetic, where errors are propagated exponentially.

(d) Given a quantum circuit consisting of n arbitrary binary gates, and given a certain tolerance $\epsilon > 0$. How many H and CONTROLLED-V gates are needed to approximate the given circuit within ϵ? As a function of n, is this quantity large (e.g., exponential) or small (e.g., polynomial)?

Problem 3. Recall that D_n is the set of density matrices of dimension n:

$$D_n = \{A \in \mathbb{C}^{n\times n} \mid A \text{ is hermitian positive and tr } A \leqslant 1\},$$

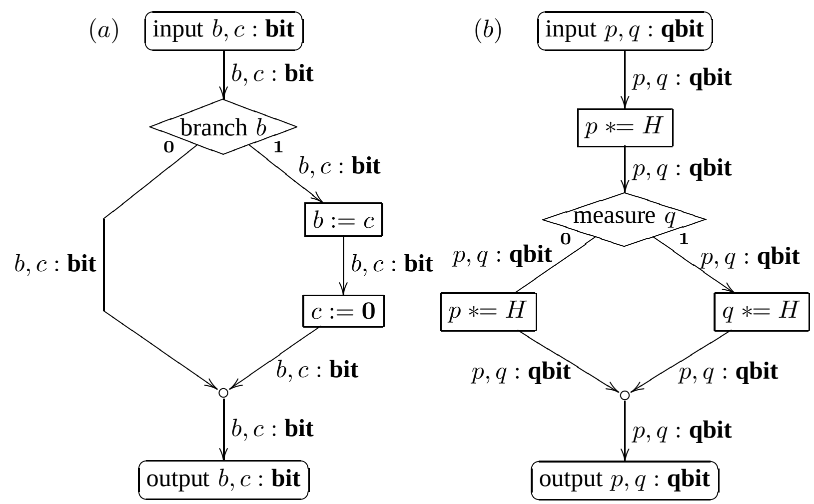
with the Löwner partial order $\sqsubseteq$ defined by $A \sqsubseteq B$ iff $B-A$ is positive. Also recall that this forms a *complete partial order*, i.e., every increasing chain $A_0 \sqsubseteq A_1 \sqsubseteq \ldots$ has a least upper bound, denoted $\bigvee^{\uparrow}_i A_i$.

(a) What are the maximal elements of D_n with respect to $\sqsubseteq$?

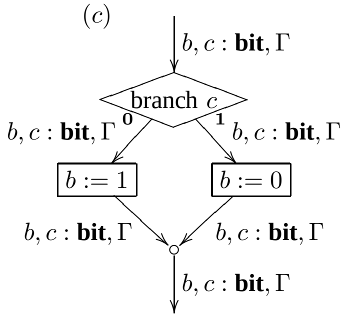
(b) Characterize the pairs of matrices for which $A \ll B$, according to the following definition:

Definition. Let A, B be two elements in a complete partially ordered set. We say that A is *way below* B, written $A \ll B$, if for all increasing chains $(A_i)_{i\in\mathbb{N}}$ with $B \sqsubseteq \bigvee^{\uparrow}_i A_i$, there exists some i with $A \sqsubseteq A_i$.

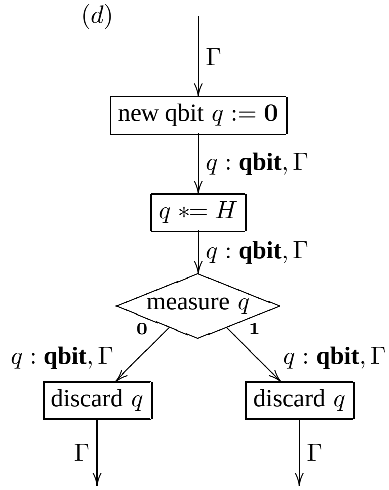
Problem 4. For each of the following flow chart procedures, calculate their denotational semantics as a superoperator. In (e), $p \oplus\!= q$ stands for $q, p \mathrel{*}= N_c$, i.e., an application of the controlled-not gate, where q is the "controlling" and p is the "controlled" qubit. In part (f), the "coin toss" refers to the flow chart from part (d). (i) is a recursive procedure, similar, but not identical, to the one given in class.



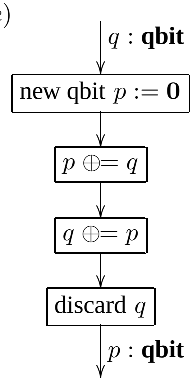
(c)



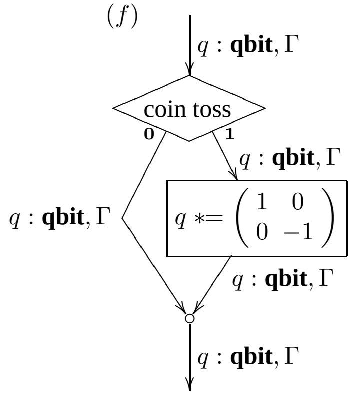
(d)



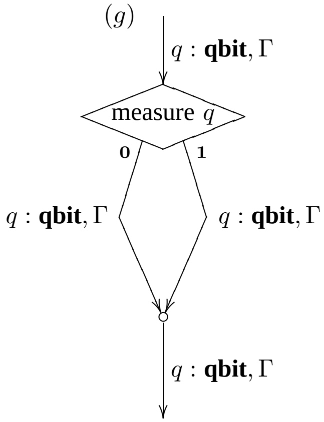
(e)



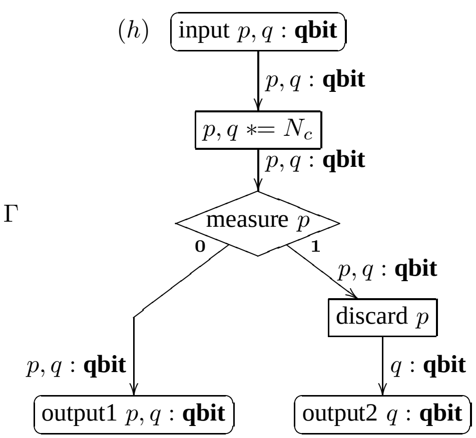
(f)



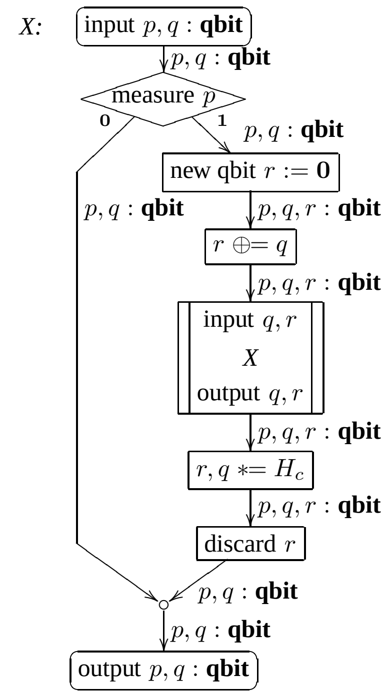
(g)



(h)



(i)



Problem 5. Which of the following functions are (1) positive, (2) completely positive, (3) superoperators? For those which are superoperators, give a Kraus representation, as well as a quantum flow chart.

(a) $F\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{2}(a+b+c+d)$

(b) $F\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 2a+b+c+2d & b+c \\ b+c & 2a+b+c+2d \end{pmatrix}$

(c) $F\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (\frac{1}{2}(a+d), \frac{1}{2}(b+c))$

(d) $F\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b/2 \\ c/2 & d \end{pmatrix}$

(e) $F(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} e & f \\ g & h \end{pmatrix}) = (\frac{1}{2}a, \begin{pmatrix} \frac{1}{4}a+e & if \\ -ig & \frac{1}{4}a+h \end{pmatrix}, d)$

Problem 6. Prove: If $F : V_\sigma \to V'_\sigma$ and $G : V_\tau \to V'_\tau$ are superoperators, then so are $F \oplus G : V_{\sigma\oplus\sigma'} \to V_{\tau\oplus\tau'}$ and $F \otimes G : V_{\sigma\otimes\sigma'} \to V_{\tau\otimes\tau'}$.

Problem 7. Recall that for linear operators $F, G : V_\sigma \to V_{\sigma'}$, we define $F \sqsubseteq G$ iff for all τ and for all positive $A \in V_{\tau\otimes\sigma}$, $(\mathrm{id}_\tau \otimes F)(A) \sqsubseteq (\mathrm{id}_\tau \otimes G)(A)$. (a) Prove: $F \sqsubseteq G$ iff $\chi_F \sqsubseteq \chi_G$. (b) Prove: In case $\sigma' = 1$, a linear operator $F : V_\sigma \to V_1$ is completely positive iff it is positive. (This is not true for general σ'). (c) Consequently, for $F, G : V_\sigma \to V_1$, we have $F \sqsubseteq G$ iff for all positive A, $F(A) \sqsubseteq G(A)$.