Agda is based on

- Type theory
- Lambda calculus
- Curry-Howard isomorphism, BHK interpretation
- Inductive Types

What are the natural numbers?

Definition The $\mathbb{N}$ of natural numbers has the following structure and properties:

- There is an element called "0": $0 : \mathbb{N}$
- There is a function called "succ": $\mathrm{succ} : \mathbb{N} \to \mathbb{N}$
- Universality: The set $\mathbb{N}$ is the "free" one with the above structure. More precisely:
  - Given any (other) set X with an element $z \in X$ and a function $s : X \to X$,
  - There exists a unique function $f : \mathbb{N} \to X$ such that
    (1) $f(0) = z$
    (2) for all $n \in \mathbb{N}$, $f(\mathrm{succ}(n)) = s(f(n))$.

To make this definition easier to understand, let's introduce some concepts.

Definition A <u>zero-successor algebra</u> is a triple (X, z, s) such that

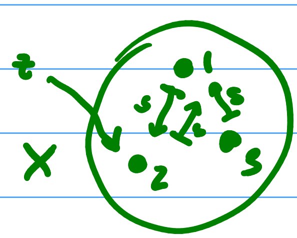
- X is a set
- $z \in X$ is an element called the "zero" of the algebra,
- $s : X \to X$ is a function called the "successor function" of the algebra.

Examples (1) The set of natural numbers $(\mathbb{N}, 0, succ)$ where $succ(n) = n+1$ is a zero-successor algebra.

(2) $(\mathbb{N}, 0, f)$ where $f(n) = n+2$. is a zero-successor algebra.

(3) Let $X = \{1,2,3\}$. Define $s : X \to X$ by $s(1) = 2$, $s(2) = 1$, $s(3) = 1$

Then $(X, 2, s)$ is a zero-successor algebra.

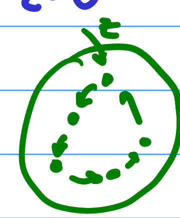


(4) Let $X = \{0,1,2,3,4\}$ Define $z = 0$ and

$s(0) = 1$ $\quad s(3) = 4$

$s(1) = 2$ $\quad s(4) = 0$

$s(2) = 3$



Definition Let (X, z, s) and (Y, z', s') be zero-successor algebras. A **homomorphism** (of zero-successor algebras) is a function

$$f : X \to Y$$

such that $\quad f(z) = z'$

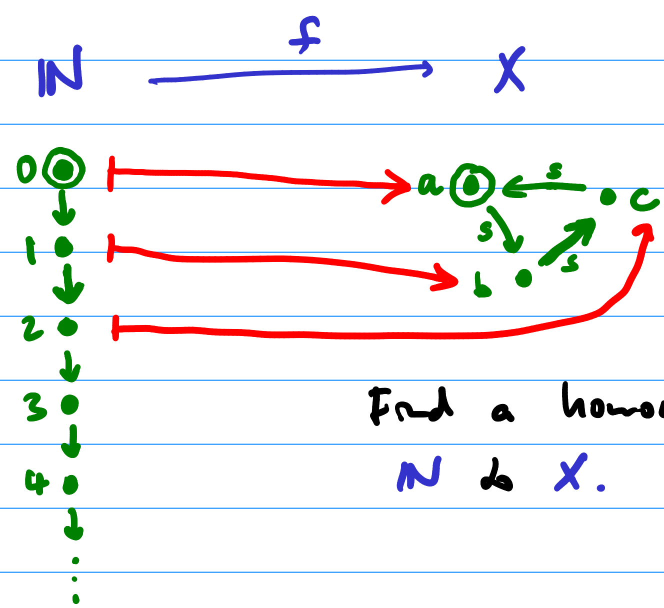
and $\quad f(s(x)) = s'(f(x))$

For category theorists, we can also write these equations as commutative diagrams:

$$\begin{tikzcd} & 1 \arrow[dl, "z"'] \arrow[dr, "z'"] & \\ X \arrow[rr, "f"'] & & Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "s"'] & Y \arrow[d, "s'"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Example Consider $(\mathbb{N}, 0, \text{succ})$ and (X, z, s)

where $X = \{a, b, c\}$, $\quad z = a$, $\quad s(a) = b$
$s(b) = c$
$s(c) = a$.



$f(0) = a$

$f(1) = f(\text{succ}(0))$
$= s(f(0))$
$= s(a)$
$= b$

$f(2) = f(\text{succ}(1))$
$= s(f(1))$
$= s(b) = c$

...

Find a homomorphism from $\mathbb{N}$ to X.

Theorem There exists a unique homomorphism of zero-successor algebras $f: \mathbb{N} \to X$

Proof By induction. $f(0)$ is uniquely determined. And whenever we know $f(n)$, then $f(n+1)$ is uniquely determined. $\square$

Theorem For <u>any</u> zero-successor algebra (X, z, s), there exists a unique homomorphism $f: \mathbb{N} \to X$.

Proof The same. $\square$

Definition A zs-algebra (X, z, s) is called <u>initial</u> if it has the property that for all zs-algebras (Y, z', s'), there exists a unique homomorphism $f: X \to Y$

Remark $(\mathbb{N}, 0, \text{succ})$ is an initial zs-algebra.

Theorem Any two initial zs-algebras are isomorphic.

Proof Let (X, z, s) and (Y, z', s') be zs-algebras, and assume both X and Y are initial.

(a) Since X is initial, there exists a unique homomorphism $f: X \to Y$.

(b) Since Y is initial, there exists a unique homomorphism $g: Y \to X$.

(c) Then $g \circ f : X \to X$

is a homomorphism.

But $id : X \to X$

is also a homomorphism.

$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \end{tikzcd}$$

But X is <u>initial</u>. Therefore <u>$g \circ f = id$</u>

(d) Similarly, <u>$f \circ g = id$</u>

(e) So f, g are inverses, so

<u>isomorphisms</u>. So (X, z, s) and (Y, z', s')

are isomorphic. $\square$

<u>Theorem</u> $(\mathbb{N}, 0, succ)$ are (up to isomorphism)

the unique initial zero-successor algebra.

$\square$

<u>In Agda</u> When we write down a definition

of a data type

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data ℕ : Set where
  zero : ℕ
  succ : ℕ → ℕ
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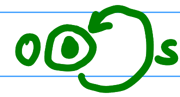
it is automatically understood that $\mathbb{N}$ is the

<u>initial</u> such algebra. This is called an

<u>inductive datatype</u>

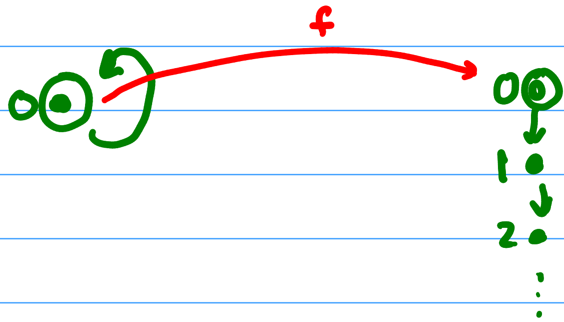
<u>Question:</u> Is (X, z, s) initial, where

$X = \{0\}$, $z = 0$, $s(0) = 0$?



No, because there exists no homomorphism into $\mathbb{N}$:

$$(X, z, s) \xrightarrow{f} (\mathbb{N}, 0, \mathrm{succ}).$$



$f(0) = 0$

$f(s(0)) = 0$

$\mathrm{succ}(f(0)) = 1$

<u>Not</u> a homo.