A quick interpretation of constructive logic

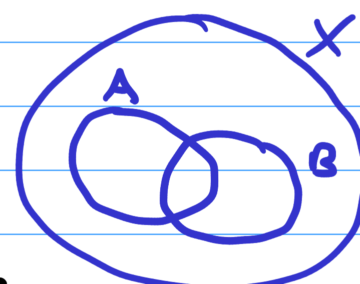
I. Classical logic

1. • **Truth tables**: Every propositional variable is interpreted as "T" or "F".

• Value of compound formulas is given by truth tables.

A	B	$A \vee B$
T	T	T
T	F	T
F	T	T
F	F	F

2. – **Venn diagrams** Given a set X called the "universe".



– Every propositional variable is assigned a subset of X.

– The logical connectives are interpreted as operations on sets:

A and B = $A \cap B$ "intersection"

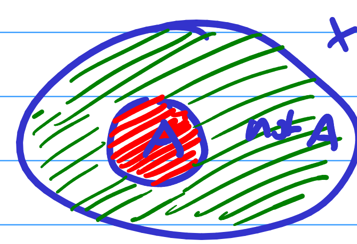
A or B = $A \cup B$ "union"

not A = $X - A = A^c$ "complement"

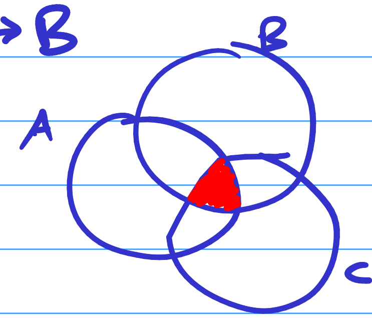
– A proposition P is a tautology or valid if $P = X$.

Ex. $A \vee \neg A$

$A \cup A^c = X$.



$(A \cap B) \cap C \to B$

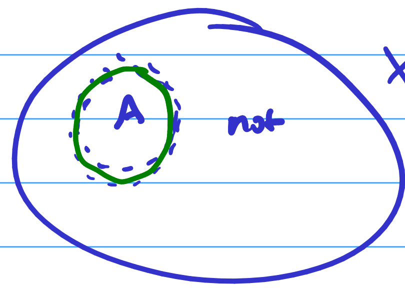


is valid because

$(A \cap B) \cap C \subseteq B$.

II. Constructive logic

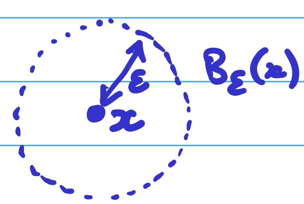
Same idea, but use open sets.



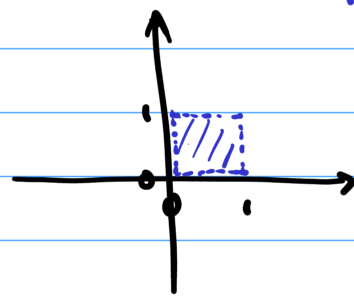
A model of intuitionistic (propositional) logic
[Stone 1937, Tarski 1938]

__Definition__ A subset A of $\mathbb{R}^2$ is called __open__ if for every $x \in A$, there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq A$.

Here $B_\varepsilon(x) = \{ y \in \mathbb{R}^2 \mid \|x - y\| < \varepsilon \}$ is the ε-neighborhood of x.



Examples Let $A = \{(x,y) \mid 0<x<1 \text{ and } 0<y<1\}$



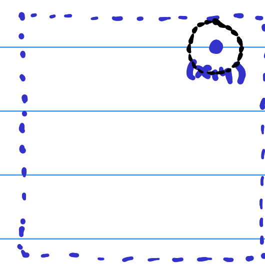
Then A is open.

Proof: Consider any $(x,y) \in A$.

Then $0<x<1$ and $0<y<1$.

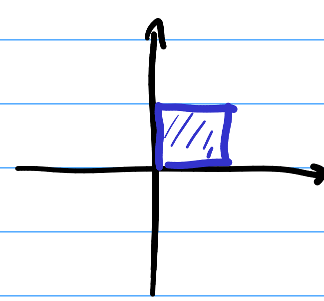
Let $\varepsilon = \min\{x, 1-x, y, 1-y\}$

Then $B_\varepsilon(x,y) \subseteq A$

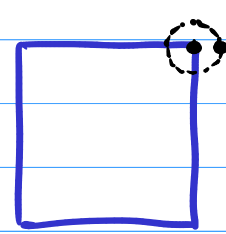


Let $B = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le 1\}$

Then B is not open.



Proof: E.g. $(1,1) \in B$, but there is no $\varepsilon > 0$ such that $B_\varepsilon(1,1) \subseteq B$.



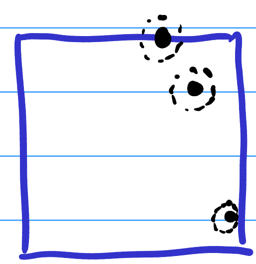
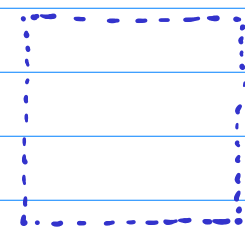
Definition Let $A \subseteq \mathbb{R}^2$ be any subset of $\mathbb{R}^2$. The interior of A, in symbols A°, is defined as follows:

$$A^\circ = \{ x \in A \mid \exists \varepsilon > 0.\ B_\varepsilon(x) \subseteq A \}.$$

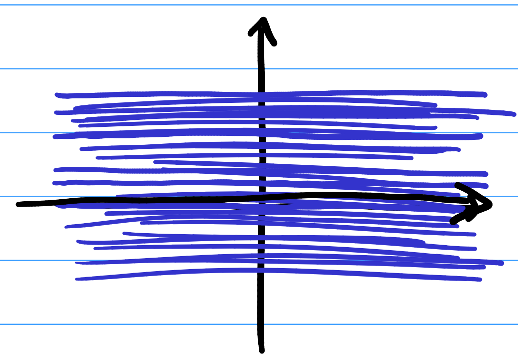
Equivalently, some point x is in the interior of A iff A contains some ε-neighborhood of x.

Example Let $B = \{ (x,y) \mid 0 \le x \le 1,\ 0 \le y \le 1 \}$.

Then $B^\circ = \{ (x,y) \mid 0 < x < 1,\ 0 < y < 1 \}$.

$B =$  $B^\circ =$ 

Example Let $C = \{ (x,y) \mid x \in \mathbb{R},\ y \in \mathbb{Q} \}$



Is C open?

No. For example $(0,0) \in C$.

But for all $\varepsilon > 0$, there exists some irrational number $y \in (0, \varepsilon)$.

Then $(0,y) \in B_\varepsilon(0,0)$ but $(0,y) \notin C$.

The interior of C is $C^{\circ} = \emptyset$.

<u>Proposition</u> If $A \subseteq \mathbb{R}^2$ is any set, then A° is open.

<u>Proof</u>: Exercise.

<u>The Stone-Tarski model of intuitionistic logic</u>

Consider the language with propositional symbols
$A_1, A_2, A_3, \dots$
and with the usual logical connectives
$\wedge, \vee, \neg, \to, T, F$

So in other words, the propositions of the logic are given by the grammar

$$A, B ::= A_i \mid (A \wedge B) \mid (A \vee B) \mid \neg A \mid (A \to B) \mid T \mid F$$

<u>Definition</u> An <u>interpretation</u> of the logic is a function σ that assigns to each A_i an open subset $\sigma(A_i) \subseteq \mathbb{R}^2$.

Moreover, we extend the interpretation to all propositions as follows:

$$\bar{\sigma}(A_i) = \sigma(A_i)$$
$$\bar{\sigma}(A \wedge B) = \bar{\sigma}(A) \cap \bar{\sigma}(B)$$
$$\bar{\sigma}(A \vee B) = \bar{\sigma}(A) \cup \bar{\sigma}(B)$$

$$\bar{\sigma}(T) = \mathbb{R}^2$$
$$\bar{\sigma}(F) = \emptyset$$
$$\bar{\sigma}(\neg A) = \left(\left(\bar{\sigma}(A)\right)^c\right)^\circ$$

This is the interior of the complement of $\bar{\sigma}(A)$. Note that the complement of an open set may not be open.

$$\bar{\sigma}(A \to B) = \left(\bar{\sigma}(A)^c \cup \bar{\sigma}(B)\right)^\circ$$

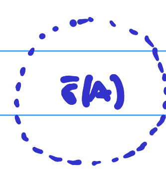
<u>Note</u>: The intersection and the union of two open sets are open. So all the sets defined above are open.

<u>Proposition</u> If a proposition A is intuitionistically valid, then for all interpretations $\bar{\sigma}$, we have $\bar{\sigma}(A) = \mathbb{R}^2$.

The converse is also true, i.e. if A is not intuitionistically valid, then there exists some interpretation such that $\bar{\sigma}(A) \neq \mathbb{R}^2$.

<u>Examples</u>

1. $A \to A$



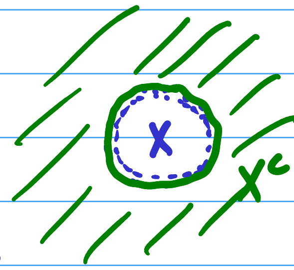
$$\bar{\sigma}(A \to A) = \left(\bar{\sigma}(A)^c \cup \bar{\sigma}(A)\right)^\circ$$
$$= (\mathbb{R}^2)^\circ$$
$$= \mathbb{R}^2.$$

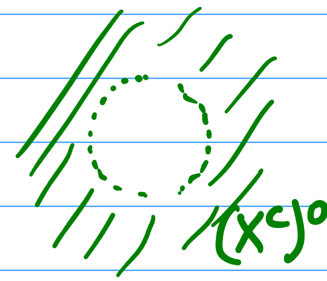
2. $A \vee \neg A$

$\bar{\sigma}(A) = X = B_1(0,0)$
$= \{(x,y) \mid x^2+y^2<1\}$

$\bar{\sigma}(A \vee \neg A) = \bar{\sigma}(A) \cup \bar{\sigma}(\neg A)$
$= \bar{\sigma}(A) \cup (\bar{\sigma}(A)^c)^o$
$= X \cup (X^c)^o$

$X = \{(x,y) \mid x^2+y^2<1\}$
$X^c = \{(x,y) \mid x^2+y^2 \geq 1\}$
$(X^c)^o = \{(x,y) \mid x^2+y^2>1\}$





$X \cup (X^c)^o = \{(x,y) \mid x^2+y^2 \neq 1\}$
$= \mathbb{R}^2 - \{(x,y) \mid x^2+y^2=1\}$
$\neq \mathbb{R}^2.$

$X \cup (X^c)^o =$

$= \mathbb{R}^2 -$ circle.

3. $\neg\neg A$ vs. A.

<u>Example</u> 1 $\bar{\sigma}(A) = X = \{(x,y) \mid x^2+y^2<1\}$
$X^c = \{(x,y) \mid x^2+y^2 \geq 1\}$
$\bar{\sigma}(\neg A)$ $(X^c)^o = \{(x,y) \mid x^2+y^2>1\}$
$((X^c)^o)^c = \{(x,y) \mid x^2+y^2 \leq 1\}$
$\bar{\sigma}(\neg\neg A) = (((X^c)^o)^c)^o = \{(x,y) \mid x^2+y^2<1\}$

Exple 2 $\bar{\sigma}(A) = X = \{(x,y) \mid x \in \mathbb{R},\ y \neq 0\}$

$X^c = \{(x,y) \mid x \in \mathbb{R},\ y = 0\}$

$\bar{\sigma}(\neg A) = (X^c)^o = \emptyset$

$((X^c)^o)^c = \mathbb{R}^2$

$\bar{\sigma}(\neg\neg A) = (((X^c)^o)^c)^o = \mathbb{R}^2$

$\bar{\sigma}(A) \neq \bar{\sigma}(\neg\neg A).$