MATH 582, INTRODUCTION TO SET THEORY, WINTER 1999

Answers to Problem Set 3 (Revised)

Problem 3.8 Consider any set $\mathcal{A}$. Then

$$\begin{array}{rcll} x \in \operatorname{dom} \bigcup \mathcal{A} & \iff & (\exists y)\langle x,y\rangle \in \bigcup \mathcal{A} & \text{(by definition of dom)} \\ & \iff & (\exists y)(\exists R \in \mathcal{A})\langle x,y\rangle \in R & \text{(by definition of union)} \\ & \iff & (\exists R \in \mathcal{A})(\exists y)\langle x,y\rangle \in R & \\ & \iff & (\exists R \in \mathcal{A}) x \in \operatorname{dom} R & \text{(by definition of dom)} \\ & \iff & x \in \bigcup\{\operatorname{dom} R \mid R \in \mathcal{A}\} & \text{(by definition of union)} \end{array}$$

The proof for "ran" is similar.

Problem 3.9 If we replace the union operation by intersection in the previous problem (assuming that $\mathcal{A}$ is non-empty), the corresponding result is not true. For a simple example, let $\mathcal{A} = \{\{\langle a,b\rangle\},\{\langle a,c\rangle\}\}$, where $b \neq c$. Then $\operatorname{dom} \bigcap \mathcal{A} = \operatorname{dom} \emptyset = \emptyset$, whereas $\bigcap\{\operatorname{dom} R \mid R \in \mathcal{A}\} = \bigcap\{\{a\}\} = \{a\}$. However, the inclusion from left to right is still valid, namely $\operatorname{dom} \bigcap \mathcal{A} \subseteq \bigcap\{\operatorname{dom} R \mid R \in \mathcal{A}\}$. Compare the proof to that of Problem 3.8:

$$\begin{array}{rcll} x \in \operatorname{dom} \bigcap \mathcal{A} & \iff & (\exists y)\langle x,y\rangle \in \bigcap \mathcal{A} & \text{(by definition of dom)} \\ & \iff & (\exists y)(\forall R \in \mathcal{A})\langle x,y\rangle \in R & \text{(by definition of intersection)} \\ & \Rightarrow & (\forall R \in \mathcal{A})(\exists y)\langle x,y\rangle \in R & \text{(*)} \\ & \iff & (\forall R \in \mathcal{A}) x \in \operatorname{dom} R & \text{(by definition of dom)} \\ & \iff & x \in \bigcap\{\operatorname{dom} R \mid R \in \mathcal{A}\} & \text{(by definition of intersection)} \end{array}$$

Notice that in the step marked (*), the converse implication does not hold: $\exists y \forall R \ldots$ implies $\forall R \exists y \ldots$, but not the other way around.

Problem 3.11 Assume that F and G are functions with $\operatorname{dom} F = \operatorname{dom} G$. Assume that $F(x) = G(x)$ for all x in the common domain. We claim that $F = G$. We will show this by proving $F \subseteq G$ and $G \subseteq F$ (recall that functions are sets, namely certain sets of ordered pairs). So suppose $z \in F$. Then, since F is a function, z is an ordered pair, i.e., $z = \langle x,y\rangle$ for some x,y. By definition of $F(x)$, we have $F(x) = y$. Since x is in the domain of F, we have $G(x) = F(x)$ by hypothesis, and thus $G(x) = y$. This implies $z = \langle x,y\rangle \in G$ by definition of $G(x)$. This proves $F \subseteq G$; the converse follows by a similar argument.

Problem 3.15 Let $\mathcal{A}$ be a set of functions such that for any $f,g \in \mathcal{A}$, either $f \subseteq g$ or $g \subseteq f$. We claim that $\bigcup \mathcal{A}$ is a function. First, notice that each element of $\bigcup \mathcal{A}$ is an ordered pair, and thus $\bigcup \mathcal{A}$ is a relation. To prove that it is a function, consider any $\langle x,y\rangle$ and $\langle x,y'\rangle$ in $\bigcup \mathcal{A}$. We have to show that $y = y'$. First, by definition of union, we know that $\langle x,y\rangle \in f$ and $\langle x,y'\rangle \in g$ for some $f,g \in \mathcal{A}$. By hypothesis, we know that either $f \subseteq g$ or $g \subseteq f$. We do a case distinction: if $f \subseteq g$, then $\langle x,y\rangle \in g$, which implies $y = y'$, since g is a function. Similarly, if $g \subseteq f$, then $\langle x,y'\rangle \in f$, which again implies $y = y'$, since f is a function. In either case, $y = y'$, and we are done.

Problem 3.29 Given $f : A \to B$, define $G : B \to \mathscr{P}A$ by $G(b) = \{x \in A \mid f(x) = b\}$. Assuming that f is onto B, we want to show that G is one-to-one. So consider $b,b' \in B$ with $G(b) = G(b')$. We have to show $b = b'$. Since f is onto, we know that there is some $a \in A$ with $f(a) = b$. It follows, by definition of G, that $a \in G(b)$, and thus, $a \in G(b')$ by hypothesis. Using the definition of G again, this in turns implies $f(a) = b'$, and thus $b = f(a) = b'$; we are done.

The converse does not in general hold. In other words, it is possible for G to be one-to-one even if f is not onto. To be precise: if there is at most one element of B that is not in the range of f, then G will still be one-to-one. Proof: Assume that there is at most one element of B that is not in the range of f. Suppose $G(b) = G(b')$. If b is in the range of f, then $b = b'$ follows as above; similarly if b' is in the range of f. The only case left is if neither b nor b' is in the range of f; in this case, $b = b'$ by hypothesis.

However, if there are at least two different elements of B that are not in the range of f, then G will not be one-to-one: Let b,b' be such different elements, then $G(b) = G(b') = \emptyset$ but $b \neq b'$, and hence G is not one-to-one.

Problem 3.30 Let $F : \mathscr{P}A \to \mathscr{P}A$ have the monotonicity property $X \subseteq Y \in \mathscr{P}A \Rightarrow F(X) \subseteq F(Y)$. Define

$$B = \bigcap\{X \subseteq A \mid F(X) \subseteq X\} \qquad \text{and} \qquad C = \bigcup\{X \subseteq A \mid X \subseteq F(X)\}.$$

Note that $F(A) \subseteq A$, and thus the set $\{X \subseteq A \mid F(X) \subseteq X\}$ is non-empty, making the above intersection well-defined.

(a) Let $\mathscr{B}$ be the set $\{X \subseteq A \mid F(X) \subseteq X\}$, so that $B = \bigcap \mathscr{B}$. We claim that $F(B) = B$.

- First, we claim that $F(B) \subseteq B$. So take any $z \in F(B)$. To show that $z \in B$, consider an arbitrary $X \in \mathscr{B}$; we have to show that $z \in X$. By definition of B, we know that $B \subseteq X$, and hence by monotonicity, $F(B) \subseteq F(X)$, which implies $z \in F(X)$. On the other hand, since $X \in \mathscr{B}$, we know that $F(X) \subseteq X$, and thus $z \in X$. Since $X \in \mathscr{B}$ was arbitrary, this proves $z \in B$, and we have $F(B) \subseteq B$ as desired.
- To prove the converse, $B \subseteq F(B)$, we use the fact that we already know $F(B) \subseteq B$. By monotonicity, this implies $F(F(B)) \subseteq F(B)$, and hence, we have $F(B) \in \mathscr{B}$. It follows that $B = \bigcap \mathscr{B} \subseteq F(B)$.

Now, let $\mathscr{C}$ be the set $\{X \subseteq A \mid X \subseteq F(X)\}$, so that $C = \bigcup \mathscr{C}$. We claim that $C = F(C)$.

- First, we claim that $C \subseteq F(C)$. So take any $z \in C$. Then there exists some $X \in \mathscr{C}$ such that $z \in X$. By definition of $\mathscr{C}$, we know that $X \subseteq F(X)$, and thus $z \in F(X)$. Also, since $C = \bigcup \mathscr{C}$, we know that $X \subseteq C$, and hence by monotonicity, $F(X) \subseteq F(C)$. It follows that $z \in F(C)$, as desired.
- To prove the converse, $F(C) \subseteq C$, we use the fact that we already know $C \subseteq F(C)$. By monotonicity, this implies $F(C) \subseteq F(F(C))$, and hence, we have $F(C) \in \mathscr{C}$. It follows that $F(C) \subseteq \bigcup \mathscr{C} = C$.

(b) Suppose $F(X) = X$. Then $X \in \mathscr{B}$, and hence $B = \bigcap \mathscr{B} \subseteq X$. Similarly, $X \in \mathscr{C}$, and hence $X \subseteq \bigcup \mathscr{C} = C$.

Homework for Feb. 5:

1. We gave two definitions of linear order in class, one in terms of $<$ and one in terms of $\leqslant$. Show that they are equivalent, i.e., show that if $\langle A, <\rangle$ is a linear order in the first sense, and if one defines

$$x \leqslant y :\Longleftrightarrow x < y \vee x = y,$$

then $\langle A, \leqslant\rangle$ is a linear order in the second sense; and if $\langle A, \leqslant\rangle$ is a linear order in the second sense, and if one defines

$$x < y :\Longleftrightarrow x \leqslant y \wedge x \neq y,$$

then $\langle A, <\rangle$ is a linear order in the first sense.

2. Prove that a relation E on A is an equivalence relation if and only if, for all $x, y \in A$,

$$xEy \iff \forall z \in A(xEz \Rightarrow yEz).$$

3. Suppose R and Q are equivalence relations on sets A and B respectively, and $f : A \to B$ is a function. Prove that there exists a function $g : A/R \to B/Q$ satisfying $g([x]_R) = [f(x)]_Q$ for all $x \in A$, if and only if xRy implies $f(x)\, Q\, f(y)$, for all $x, y \in A$.

Chapter 3, Problems 34, 36, 41, 44, 45. Chapter 6, Problem 22 [or 18]. Chapter 7, Problem 1.