MATH 582, INTRODUCTION TO SET THEORY, WINTER 1999

Answers to Problem Set 7

Problem from Class. Using the replacement axiom, give an alternative proof that $A \times B = \{\langle a, b\rangle \mid a \in A \wedge b \in B\}$ is a set, for sets A and B.

The trick is that we must apply the replacement axiom in each component separately. First, for each $b \in B$, we apply the replacement axiom to the set A and the formula $\phi(x, y) \equiv (y = \langle x, b\rangle)$ to obtain the set $A_b = \{\langle a, b\rangle \mid a \in A\}$. Then, we can apply the replacement axiom again, this time to the set B and the formula $\psi(x, y) \equiv (y = A_x)$, to obtain the set $C = \{A_b \mid b \in B\}$. Then $x \in \bigcup C$ if and only if for some $b \in B$, $x \in A_b$, if and only if for some $b \in B$ and some $a \in A$, $x = \langle a, b\rangle$. Thus, $\bigcup C = A \times B$ is the desired product.

Problem 7.4 First, notice that the definition of R is equivalent to

$$mRn \iff \phi(m) <_L \phi(n),$$

where $\phi : P \to P \times P$ is the function $\phi(n) = \langle f(n), n\rangle$, and $<_L$ is the lexicographic order on $P \times P$. Notice that ϕ is one-to-one.

We prove that R is a well-order by showing two things: (a) $<_L$ is a well-order on $P \times P$, and (b) well-orders are reflected by one-to-one functions. The last statement means that if $\langle W, <\rangle$ is a well-order and $\phi : B \to W$ is a one-to-one function, then the relation R that is defined on B by $xRy \iff \phi(x) < \phi(y)$ is a well-order on B.

Clearly, from these two statements it follows that R is a well-order.

(a) We know from Problem 3.45 that the lexicographic order $<_L$ is a linear order on $P \times P$. To show that $<_L$ is a well-order, take any non-empty subset A of $P \times P$. Let $A_0 = \{m \in P \mid \exists n.\langle m, n\rangle \in A\}$. Then A_0 is non-empty, thus it has a least element m_0, by the well-order property of P. Now let $A_1 = \{n \in P \mid \langle m_0, n\rangle \in A\}$. Then A_1 is non-empty, thus it has a least element n_0. We claim that $\langle m_0, n_0\rangle$ is the least element of $\langle A, <_L\rangle$. Clearly, by construction, $\langle m_0, n_0\rangle \in A$. Consider any other $\langle m, n\rangle \in A$. Then $m \in A_0$, and thus $m_0 \leqslant m$ by leastness of m_0. There are two cases: either $m_0 < m$, in which case $\langle m_0, n_0\rangle <_L \langle m, n\rangle$ by definition of the lexicographic order. Or else, $m_0 = m$. In the latter case, we have $n \in A_1$, and by the leastness of n_0, it follows that $n_0 \leqslant n$. Again, by the definition of the lexicographic order, this implies $\langle m_0, n_0\rangle \leqslant_L \langle m, n\rangle$, showing that $\langle m_0, n_0\rangle$ is the least element of A. Thus, $<_L$ is a well-order on $P \times P$.

(b) Suppose $\langle W, <\rangle$ is a well-order and $\phi : B \to W$ is a one-to-one function. Define a relation R on B by $xRy \iff \phi(x) < \phi(y)$. One easily sees that this relation is irreflexive, transitive, and connected: thus it is a linear order. Now if A is a non-empty subset of B, then $\phi[\![A]\!]$ is a non-empty subset of W, thus it has a least element $\phi(x)$. This means that for all $y \in A$, $\phi(x) \leqslant \phi(y)$, hence xRy. Thus, x is a least element of A, showing that R is a well-order.

The claim follows. Actually, the well-order R resembles that shown in Fig. 45(d).

Problem 7.5 Suppose $x \leqslant f(x)$ does not hold for all $x \in A$; then there is a least $x \in A$ such that $f(x) < x$. By hypothesis, this implies $f(f(x)) < f(x)$, i.e. $f(y) < y$, where $y = f(x)$. But $y < x$, contradicting the leastness of x.

Problem 7.7 Let C be a fixed set, and let $\gamma(x, y)$ be the formula

$$y = C \cup \bigcup\bigcup \operatorname{ran} x.$$

Clearly, for every set x, there exists a unique y with $\gamma(x, y)$. Thus, we can apply transfinite recursion to obtain a function F with domain ω, such that for all $n \in \omega$, $\gamma(F \restriction \operatorname{seg} n, F(n))$, which is to say,

$$\begin{aligned} F(n) &= C \cup \bigcup\bigcup \operatorname{ran}(F \restriction \operatorname{seg} n) \\ &= C \cup \bigcup\bigcup F[\![\operatorname{seg} n]\!]. \end{aligned}$$

(a)

$$\begin{aligned} F(0) &= C \cup \bigcup\bigcup \emptyset = C \\ F(1) &= C \cup \bigcup\bigcup \{F(0)\} = C \cup \bigcup C \\ F(2) &= C \cup \bigcup\bigcup \{F(0), F(1)\} = C \cup \bigcup (C \cup \bigcup C) \\ &= C \cup \bigcup C \cup \bigcup\bigcup C \text{ (by Problem 2.21)} \end{aligned}$$

Our best guess is that $F(n) = C \cup \bigcup C \cup \ldots \cup \bigcup^n C$.

(b) Suppose $a \in F(n)$. Then

$$\begin{aligned} a &\subseteq \bigcup F(n) && \text{by Problem 2.3} \\ &= \bigcup\bigcup \{F(n)\} \\ &\subseteq C \cup \bigcup\bigcup F[\![\operatorname{seg} n^+]\!] && \text{because } \{F(n)\} \subseteq F[\![\operatorname{seg} n^+]\!] \\ &= F(n^+). \end{aligned}$$

(c) Let $\bar{C} = \bigcup \operatorname{ran} F = \bigcup_{n\in\omega} F(n)$. Then $C = F(0) \subseteq \bar{C}$, and $\bar{C}$ is transitive: if $a \in \bar{C}$, then $a \in F(n)$ for some $n \in \omega$, and thus $a \subseteq F(n^+)$ by (b), which implies $a \subseteq \bar{C}$.

Moreover, one can show that $\bar{C}$ is actually the *smallest* transitive set containing C. To prove this, one first proves that if C is already a transitive set, then $F(n) = C$, for all n, and thus $\bar{C} = C$. One can prove this claim by transfinite induction on n: for the induction hypothesis, assume that $F(x) = C$ has already been shown for all $x \in \operatorname{seg} n$. Then $F[\![\operatorname{seg} n]\!] \subseteq \{C\}$, and equality holds if and only if $n \neq 0$. We have $F(n) = C \cup \bigcup\bigcup F[\![\operatorname{seg} n]\!] \subseteq C \cup \bigcup\bigcup\{C\} = C \cup \bigcup C = C$. The last step follows because C is transitive. On the other hand, clearly $C \subseteq F(n)$, hence $C = F(n)$ as desired.

Next, one proves that $C \subseteq D$ implies $\bar{C} \subseteq \bar{D}$; this is again shown by transfinite induction. Now it follows that if C is any set, and D is a transitive set containing C, then $\bar{C} \subseteq \bar{D} = D$. Hence $\bar{C}$ is contained in any transitive set containing C, as desired. For this reason, $\bar{C}$ is called the *transitive closure* of C.

Problem 7.8 Let $\phi(x)$ be any formula not containing the variable B. ($\phi(x)$ may contain some other variables). We want to prove the subset axiom

$$\forall A \exists B \forall x (x \in B \iff x \in A \wedge \phi(x))$$

from the other axioms. Consider the formula $\psi(x, y) \equiv (x = y \wedge \phi(y))$. Clearly, $\psi(x, y_1)$ and $\psi(x, y_2)$ implies $y_1 = y_2 = x$, and we can apply the replacement axiom schema to ψ to obtain a set B such that for all y, $y \in B$ if and only if there exists $x \in A$ such that $\psi(x, y)$. By definition of ψ, we thus have $y \in B$ if and only if $y \in A$ and $\phi(y)$, and thus B is the set that the subset axiom requires.

Problem 7.9 First, use the empty set axiom and twice the power set axiom to get the set $\mathscr{P}\mathscr{P}\emptyset = \{\emptyset, \{\emptyset\}\}$. This set has precisely two elements. Given any sets u and v, we can construct the pair set $\{u, v\}$ by the replacement axiom: Let $\psi(x, y)$ be the formula

$$(x = \emptyset \wedge y = u) \vee (x = \{\emptyset\} \wedge y = v).$$

Clearly, for each x there exists at most one y such that $\psi(x, y)$, so by replacement, there exists a set B such that $y \in B$ if and only if $\psi(x, y)$ for some $x \in \{\emptyset, \{\emptyset\}\}$, if and only if $y \in \{u, v\}$.