MATH 582, INTRODUCTION TO SET THEORY, WINTER 1999

Answers to Problem Set 9

Problem 1 Let $\mathscr{A}$ be the set of all chains of a given poset $\langle P, \leqslant\rangle$. Notice that the elements of $\mathscr{A}$ are chains in P, but we can also speak of chains in $\mathscr{A}$ (in the sense of Zorn's Lemma). To avoid confusion, we will speak of P-chains $C \subseteq P$, and of $\mathscr{A}$-chains $\mathscr{C} \subseteq \mathscr{A}$.

To apply Zorn's Lemma, we must show that $\mathscr{A}$ is closed under unions of $\mathscr{A}$-chains. So let $\mathscr{C} \subseteq \mathscr{A}$ be an $\mathscr{A}$-chain. We claim that $\bigcup\mathscr{C} \in \mathscr{A}$, i.e., that $\bigcup\mathscr{C}$ is a P-chain. Clearly, $\bigcup\mathscr{C} \subseteq P$, so we must show that for all $x, y \in \bigcup\mathscr{C}$, either $x \leqslant y$ or $y \leqslant x$. Since $x, y \in \bigcup\mathscr{C}$, we must have $x \in C$ and $y \in D$ for some $C, D \in \mathscr{C}$. Since $\mathscr{C}$ is an $\mathscr{A}$-chain, either $C \subseteq D$ or $D \subseteq C$. Assume without loss of generality that $C \subseteq D$. Then $x, y \in D$. But since $D \in \mathscr{A}$ is a P-chain, we have $x \leqslant y$ or $y \leqslant x$, as desired. Thus, $\bigcup\mathscr{C}$ is a P-chain, and $\mathscr{A}$ is closed under unions of $\mathscr{A}$-chains. It follows by Zorn's Lemma that $\mathscr{A}$ has a maximal element, i.e., there is a maximal P-chain.

Problem 2 Suppose $\langle P, \leqslant\rangle$ is a poset such that any P-chain has an upper bound in P. By Problem 1, there exists a maximal chain $C \subseteq P$. By assumption, C has an upper bound $x \in P$, i.e., there is $x \in P$ such that for all $y \in C$, $y \leqslant x$. We claim that x is a maximal element in P. For otherwise, there would exist $z \in P$ such that $x < z$. But then $C \cup \{z\}$ would be a P-chain which strictly contains C, contradicting the maximality of C.

Problem 3 The original hint given with this problem was wrong. The problem is that the given set

$$\mathscr{A} = \{R \subseteq A \times A \mid R \text{ is a well-order on some subset of } A\}$$

is not closed under unions of chains, because a union of a chain of well-orders is not necessarily a well-order. For instance, the usual order $<$ on the integers $\mathbb{Z}$ is the union of a chain of well-orders on subsets of $\mathbb{Z}$, namely the subsets of the form $\{x \in \mathbb{Z} \mid x \geqslant a\}$ for $a \in \mathbb{Z}$.

So we need to modify the argument somewhat. Instead of considering the inclusion ordering on $\mathscr{A}$, we will consider the more restrictive ordering of "being an initial piece". We will then use the poset version of Zorn's Lemma (from Problem 2).

For a relation R, recall that the ***field*** of R, $\operatorname{fld} R$, was defined to be $\operatorname{dom} R \cup \operatorname{ran} R$. Thus, any $R \in \mathscr{A}$ is a well-order on $\operatorname{fld} R$. If $R, Q \in \mathscr{A}$, we say that R is an ***initial piece*** of Q, in symbols $R \leqslant_{ip} Q$, if $R \subseteq Q$ and for all $x, y \in A$,

$$yQx \text{ and } x \in \operatorname{fld} R \text{ implies } yRx. \tag{1}$$

It is easily seen that $\leqslant_{ip}$ is reflexive, anti-symmetric, and transitive, and thus $\leqslant_{ip}$ defines a partial order on $\mathscr{A}$. We claim that every $\leqslant_{ip}$-chain has an upper bound in $\mathscr{A}$. Let $\mathscr{C}$ be any such chain. Define $\bar{R} = \bigcup\mathscr{C}$. Notice that $\operatorname{fld} \bar{R} = \bigcup\{\operatorname{fld} R \mid R \in \mathscr{C}\} \subseteq A$. We claim that $\bar{R}$ is a well-order on $\operatorname{fld} \bar{R}$, and that it is an upper bound for $\mathscr{C}$ with respect to $\leqslant_{ip}$.

Irreflexivity: Consider $x \in \operatorname{fld} \bar{R}$. Since any $R \in \mathscr{C}$ is irreflexive, we have $\neg xRx$, and hence $\neg x\bar{R}x$. *Transitivity:* Consider $x, y, z \in \operatorname{fld} \bar{R}$ such that $x\bar{R}y$ and $y\bar{R}z$. Then xRy for some $R \in \mathscr{C}$, and yQz for some $Q \in \mathscr{C}$. Since $\mathscr{C}$ is a $\leqslant_{ip}$-chain, either $R \subseteq Q$ or $Q \subseteq R$; assume without loss of generality that $R \subseteq Q$. Then $xQyQz$, hence xQz by transitivity of Q, and hence $x\bar{R}z$ as desired. *Connectedness:* Consider $x, y \in \operatorname{fld} \bar{R}$. Then $x \in \operatorname{fld} R$ for some $R \in \mathscr{C}$, and $y \in \operatorname{fld} Q$ for some $Q \in \mathscr{C}$. As before, we can assume without loss of generality that $R \subseteq Q$. Then $x, y \in \operatorname{fld} Q$, and hence xQy or $x = y$ or yQx by connectedness of Q. But since $Q \subseteq \bar{R}$, this implies $x\bar{R}y$ or $x = y$ or $y\bar{R}x$.

Well-order: Consider any non-empty subset $B \subseteq \operatorname{fld} \bar{R}$. We claim that B has a least element with respect to $\bar{R}$. First, since B is non-empty, there must be some $R \in \mathscr{C}$ such that $B \cap \operatorname{fld} R$ is non-empty. Since R is a well-order, there exists a least element $x_0 \in B \cap \operatorname{fld} R$, with respect to R. We claim that x_0 is a least element of B with respect to $\bar{R}$. Assume to the contrary that there was some $y \in B$ with $y\bar{R}x_0$. Then yQx_0 for some $Q \in \mathscr{C}$. Since we cannot have yRx_0 (by leastness of x_0), it follows that $Q \not\subseteq R$, and thus $Q \not\leqslant_{ip} R$. But since $\mathscr{C}$ is a chain, it must then be the case that $R \leqslant_{ip} Q$. But $x_0 \in \operatorname{fld} R$ and yQx_0, which implies yRx_0 by (1), contradicting the leastness of x_0. Hence, $y\bar{R}x_0$ is impossible, and x_0 is least in B with respect to $\bar{R}$.

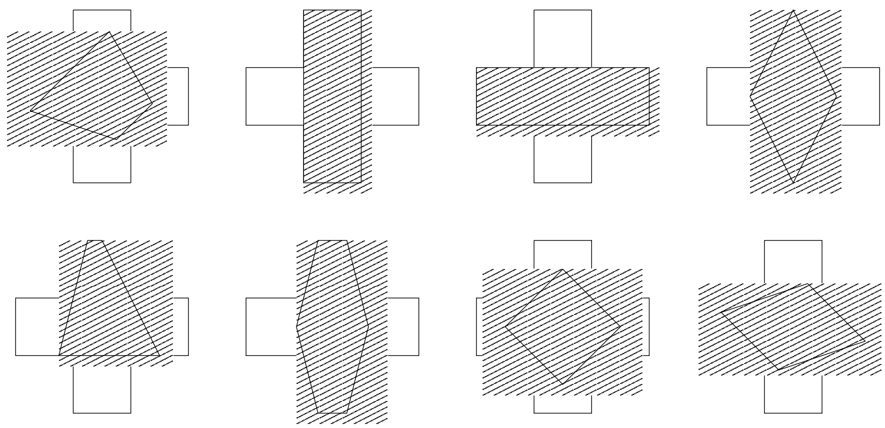
Thus, $\bar{R}$ is a well-order, and since $\bar{R} = \bigcup\mathscr{C} \subseteq A \times A$, it follows that $\bar{R} \in \mathscr{A}$. Next, we show that $\bar{R}$ is an upper bound for $\mathscr{C}$. For any $R \in \mathscr{C}$, we must show $R \leqslant_{ip} \bar{R}$. First, since $\bar{R} = \bigcup\mathscr{C}$, it is clear that $R \subseteq \bar{R}$. To show (1), consider any $x, y \in A$ such that $y\bar{R}x$ and $x \in \operatorname{fld} R$. Then yQx for some $Q \in \mathscr{C}$. Since $\mathscr{C}$ is a chain, either $R \leqslant_{ip} Q$ or $Q \leqslant_{ip} R$. In the first case, yRx by (1), applied to R and Q. In the second case, $Q \subseteq R$, and thus also yRx. This proves $R \leqslant_{ip} \bar{R}$.

We have shown that any chain in $\langle \mathscr{A}, \leqslant_{ip} \rangle$ has an upper bound. Now we can apply Zorn's Lemma for posets (Problem 2) to conclude that $\mathscr{A}$ has a maximal element R with respect to $\leqslant_{ip}$. We claim that $\operatorname{fld} R = A$. For suppose otherwise. Then there is some $x \in A$ such that $x \notin \operatorname{fld} R$. Consider $R' = R \cup (\operatorname{fld} R \times \{x\})$. It is easily seen that R' is a well-order and that $R <_{ip} R'$, contradicting the maximality of R. Hence, $\operatorname{fld} R = A$, and thus R is a well-order on A. This proves the well-ordering theorem.

Problem 4

(a) Let $\mathscr{A}$ be the set of convex subsets of X. We claim that $\mathscr{A}$ is closed under unions of chains. So let $\mathscr{C}$ be a chain of convex subsets of X, and let $Y = \bigcup \mathscr{C}$. We have to show that Y is convex. So take $u, v \in Y$. By definition of Y, we have $u \in A$ and $v \in B$ for some $A, B \in \mathscr{C}$. Since $\mathscr{C}$ is a chain, either $A \subseteq B$ or $B \subseteq A$. Let us say, without loss of generality, that $A \subseteq B$. Then $u, v \in B$, and since B is convex, the line segment connecting u and v is contained in B. But $B \subseteq Y$, and thus the line segment connecting u and v is also contained in Y. Since $u, v \in Y$ were arbitrary, this shows that $Y = \bigcup \mathscr{C}$ is convex. Thus, $\mathscr{C}$ is closed under unions of chains. It follows by Zorn's Lemma that there is a maximal $M \in \mathscr{C}$, i.e. a maximal convex subset of X.

(b) Here are some examples of maximal convex subsets of the given set X. As you can see, such sets are not at all unique.



Problem 5 Before we prove this, let us observe that any non-empty finite chain $\{B_1, \ldots, B_n\}$ has a maximal element. This is proved by induction on n: In case $n = 1$, this is clear. For the induction step, assume the claim holds for n and consider $\mathscr{C} = \{B_1, \ldots, B_{n+1}\}$. By induction hypothesis, $\{B_1, \ldots, B_n\}$ has a maximal element, say, B_i. Then, since $\mathscr{C}$ is a chain, either $B_{n+1} \subseteq B_i$ or $B_i \subseteq B_{n+1}$. In the first case, B_i is maximal in $\mathscr{C}$, and in the second case, B_{n+1} is maximal in $\mathscr{C}$.

Now on to Problem 5. It suffices to show that $\mathscr{A}$ is closed under unions of chains. So let $\mathscr{C} \subseteq \mathscr{A}$ be an arbitrary chain. Let $B = \bigcup \mathscr{C}$. We claim that $B \in \mathscr{A}$. By assumption, it suffices to show that every finite subset of B is a member of $\mathscr{A}$. So let $F = \{x_1, \ldots, x_n\} \subseteq B$ be an arbitrary such finite subset. Since $x_1, \ldots, x_n \in \bigcup \mathscr{C}$, there must be sets $B_1, \ldots, B_n \in \mathscr{C}$ such that $x_i \in B_i$ for $i = 1 \ldots n$. Since $\mathscr{C}$ is a chain, the set $\{B_1, \ldots, B_n\} \subseteq \mathscr{C}$ is a finite chain, and thus it has a maximal element B_i by our above observation. Then $x_1, \ldots, x_n \in B_i$, and thus $F \subseteq B_i$. However, $B_i \in \mathscr{A}$, and thus also $F \in \mathscr{A}$, by assumption on $\mathscr{A}$. Since F was an arbitrary finite subset of B, it follows that $B \in \mathscr{A}$, and thus $\mathscr{A}$ is closed under unions of chains. By Zorn's Lemma, $\mathscr{A}$ has a maximal element.