Lemma 4.6. *(a) For all M, M', if $M \to_{\beta\eta} M'$ then $M \triangleright M'$.*

(b) For all M, M', if $M \triangleright M'$ then $M \twoheadrightarrow_{\beta\eta} M'$.

(c) $\twoheadrightarrow_{\beta\eta}$ is the reflexive, transitive closure of $\triangleright$.

Proof. (a) First note that we have $P \triangleright P$, for any term P. This is easily shown by induction on P. We now prove the claim by induction on a derivation of $M \to_{\beta\eta} M'$. Please refer to pages 14 and 24 for the rules that define $\to_{\beta\eta}$. We make a case distinction based on the last rule used in the derivation of $M \to_{\beta\eta} M'$.

- If the last rule was (β), then $M = (\lambda x.Q)N$ and $M' = Q[N/x]$, for some Q and N. But then $M \triangleright M'$ by (4), using the facts $Q \triangleright Q$ and $N \triangleright N$.

- If the last rule was (η), then $M = \lambda x.Px$ and $M' = P$, for some P such that $x \notin FV(P)$. Then $M \triangleright M'$ follows from (5), using $P \triangleright P$.

- If the last rule was (*cong*$_1$), then $M = PN$ and $M' = P'N$, for some P, P', and N where $P \to_{\beta\eta} P'$. By induction hypothesis, $P \triangleright P'$. From this and $N \triangleright N$, it follows immediately that $M \triangleright M'$ by (2).

- If the last rule was (*cong*$_2$), we proceed similarly to the last case.

- If the last rule was (ξ), then $M = \lambda x.N$ and $M' = \lambda x.N'$ for some N and N' such that $N \to_{\beta\eta} N'$. By induction hypothesis, $N \triangleright N'$, which implies $M \triangleright M'$ by (3).

(b) We prove this by induction on a derivation of $M \triangleright M'$. We distinguish several cases, depending on the last rule used in the derivation.

- If the last rule was (1), then $M = M' = x$, and we are done because $x \twoheadrightarrow_{\beta\eta} x$.

- If the last rule was (2), then $M = PN$ and $M' = P'N'$, for some P, P', N, N' with $P \triangleright P'$ and $N \triangleright N'$. By induction hypothesis, $P \twoheadrightarrow_{\beta\eta} P'$ and $N \twoheadrightarrow_{\beta\eta} N'$. Since $\twoheadrightarrow_{\beta\eta}$ satisfies (*cong*), it follows that $PN \twoheadrightarrow_{\beta\eta} P'N'$, hence $M \twoheadrightarrow_{\beta\eta} M'$ as desired.

- If the last rule was (3), then $M = \lambda x.N$ and $M' = \lambda x.N'$, for some N, N' with $N \triangleright N'$. By induction hypothesis, $N \twoheadrightarrow_{\beta\eta} N'$, hence $M = \lambda x.N \twoheadrightarrow_{\beta\eta} \lambda x.N' = M'$ by (ξ).

- If the last rule was (4), then $M = (\lambda x.Q)N$ and $M' = Q'[N'/x]$, for some Q, Q', N, N' with $Q \triangleright Q'$ and $N \triangleright N'$. By induction hypothesis, $Q \twoheadrightarrow_{\beta\eta} Q'$ and $N \twoheadrightarrow_{\beta\eta} N'$. Therefore $M = (\lambda x.Q)N \twoheadrightarrow_{\beta\eta} (\lambda x.Q')N' \to_{\beta\eta} Q'[N'/x] = M'$, as desired.

- If the last rule was (5), then $M = \lambda x.Px$ and $M' = P'$, for some P, P' with $P \triangleright P'$, and $x \notin FV(P)$. By induction hypothesis, $P \twoheadrightarrow_{\beta\eta} P'$, hence $M = \lambda x.Px \to_{\beta\eta} P \twoheadrightarrow_{\beta\eta} P' = M'$, as desired.

(c) This follows directly from (a) and (b). Let us write R^* for the reflexive transitive closure of a relation R. By (a), we have $\to_{\beta\eta} \subseteq \triangleright$, hence $\twoheadrightarrow_{\beta\eta} = {\to_{\beta\eta}}^* \subseteq \triangleright^*$. By (b), we have $\triangleright \subseteq \twoheadrightarrow_{\beta\eta}$, hence $\triangleright^* \subseteq {\twoheadrightarrow_{\beta\eta}}^* = \twoheadrightarrow_{\beta\eta}$. It follows that $\triangleright^* = \twoheadrightarrow_{\beta\eta}$. □

We will soon prove that $\triangleright$ satisfies the diamond property. Note that together with Lemma 4.6(c), this will immediately imply that $\twoheadrightarrow_{\beta\eta}$ satisfies the Church-Rosser property.

Lemma 4.7 (Substitution)**.** *If $M \triangleright M'$ and $U \triangleright U'$, then $M[U/y] \triangleright M'[U'/y]$.*

Proof. We assume without loss of generality that any bound variables of M are different from y and from the free variables of U. The claim is now proved by induction on derivations of $M \triangleright M'$. We distinguish several cases, depending on the last rule used in the derivation:

- If the last rule was (1), then $M = M' = x$, for some variable x. If $x = y$, then $M[U/y] = U \triangleright U' = M'[U'/y]$. If $x \neq y$, then by (1), $M[U/y] = y \triangleright y = M'[U'/y]$.

- If the last rule was (2), then $M = PN$ and $M' = P'N'$, for some P, P', N, N' with $P \triangleright P'$ and $N \triangleright N'$. By induction hypothesis, $P[U/y] \triangleright P'[U'/y]$ and $N[U/y] \triangleright N'[U'/y]$, hence by (2), $M[U/y] = P[U/y]N[U/y] \triangleright P'[U'/y]N'[U'/y] = M'[U'/y]$.

- If the last rule was (3), then $M = \lambda x.N$ and $M' = \lambda x.N'$, for some N, N' with $N \triangleright N'$. By induction hypothesis, $N[U/y] \triangleright N'[U'/y]$, hence by (3) $M[U/y] = \lambda x.N[U/y] \triangleright \lambda x.N'[U'/y] = M'[U'/y]$.

- If the last rule was (4), then $M = (\lambda x.Q)N$ and $M' = Q'[N'/x]$, for some Q, Q', N, N' with $Q \triangleright Q'$ and $N \triangleright N'$. By induction hypothesis, $Q[U/y] \triangleright Q'[U'/y]$ and $N[U/y] \triangleright N'[U'/y]$, hence by (4), $(\lambda x.Q[U/y])N[U/y] \triangleright Q'[U'/y][N'[U'/y]/x] = Q'[N'/x][U'/y]$. Thus $M[U/y] = M'[U'/y]$.

- If the last rule was (5), then $M = \lambda x.Px$ and $M' = P'$, for some P, P' with $P \rhd P'$, and $x \notin FV(P)$. By induction hypothesis, $P[U/y] \rhd P'[U/y]$, hence by (5), $M[U/y] = \lambda x.P[U/y]x \rhd P'[U'/y] = M'[U'/y]$. □

A more conceptual way of looking at this proof is the following: consider any derivation of $M \rhd M'$ from axioms (1)–(5). In this derivation, replace any axiom $y \rhd y$ by $U \rhd U'$, and propagate the changes (i.e., replace y by U on the left-hand-side, and by U' on the right-hand-side of any $\rhd$). The result is a derivation of $M[U/y] \rhd M'[U'/y]$. (The formal proof that the result of this replacement is indeed a valid derivation requires an induction, and this is the reason why the proof of the substitution lemma is so long).

Our next goal is to prove that $\rhd$ satisfies the diamond property. Before proving this, we first define the *maximal parallel one-step reduct* M^* of a term M as follows:

1. $x^* = x$, for a variable.
2. $(PN)^* = P^*N^*$, if PN is not a β-redex.
3. $((\lambda x.Q)N)^* = Q^*[N^*/x]$.
4. $(\lambda x.N)^* = \lambda x.N^*$, if $\lambda x.N$ is not an η-redex.
5. $(\lambda x.Px)^* = P^*$, if $x \notin FV(P)$.

Note that M^* depends only on M. The following lemma implies the diamond property for $\rhd$.

Lemma 4.8 (Maximal parallel one-step reductions)**.** *Whenever $M \rhd M'$, then $M' \rhd M^*$.*

Proof. By induction on the size of M. We distinguish five cases, depending on the last rule used in the derivation of $M \rhd M'$. As usual, we assume that all bound variables have been renamed to avoid clashes.

- If the last rule was (1), then $M = M' = x$, also $M^* = x$, and we are done.
- If the last rule was (2), then $M = PN$ and $M' = P'N'$, where $P \rhd P'$ and $N \rhd N'$. By induction hypothesis $P' \rhd P^*$ and $N' \rhd N^*$. Two cases:
  - If PN is not a β-redex, then $M^* = P^*N^*$. Thus $M' = P'N' \rhd P^*N^* = M^*$ by (2), and we are done.
  - If PN is a β-redex, say $P = \lambda x.Q$, then $M^* = Q^*[N^*/x]$. We distinguish two subcases, depending on the last rule used in the derivation of $P \rhd P'$:
    - ∗ If the last rule was (3), then $P' = \lambda x.Q'$, where $Q \rhd Q'$. By induction hypothesis $Q' \rhd Q^*$, and with $N' \rhd N^*$, it follows that $M' = (\lambda x.Q')N' \rhd Q^*[N^*/x] = M^*$ by (4).
    - ∗ If the last rule was (5), then $P = \lambda x.Rx$ and $P' = R'$, where $x \notin FV(R)$ and $R \rhd R'$. Consider the term $Q = Rx$. Since $Rx \rhd R'x$, and Rx is a subterm of M, by induction hypothesis $R'x \rhd (Rx)^*$. By the substitution lemma, $M' = R'N' = (R'x)[N'/x] \rhd (Rx)^*[N^*/x] = M^*$.
- If the last rule was (3), then $M = \lambda x.N$ and $M' = \lambda x.N'$, where $N \rhd N'$. Two cases:
  - If M is not an η-redex, then $M^* = \lambda x.N^*$. By induction hypothesis, $N' \rhd N^*$, hence $M' \rhd M^*$ by (3).
  - If M is an η-redex, then $N = Px$, where $x \notin FV(P)$. In this case, $M^* = P^*$. We distinguish two subcases, depending on the last rule used in the derivation of $N \rhd N'$:
    - ∗ If the last rule was (2), then $N' = P'x$, where $P \rhd P'$. By induction hypothesis $P' \rhd P^*$. Hence $M' = \lambda x.P'x \rhd P^* = M^*$ by (5).
    - ∗ If the last rule was (4), then $P = \lambda y.Q$ and $N' = Q'[x/y]$, where $Q \rhd Q'$. Then $M' = \lambda x.Q'[x/y] = \lambda y.Q'$ (note $x \notin FV(Q')$). But $P \rhd \lambda y.Q'$, hence by induction hypothesis, $\lambda y.Q' \rhd P^* = M^*$.
- If the last rule was (4), then $M = (\lambda x.Q)N$ and $M' = Q'[N'/x]$, where $Q \rhd Q'$ and $N \rhd N'$. Then $M^* = Q^*[N^*/x]$, and $M' \rhd M^*$ by the substitution lemma.
- If the last rule was (5), then $M = \lambda x.Px$ and $M' = P'$, where $P \rhd P'$ and $x \notin FV(P)$. Then $M^* = P^*$. By induction hypothesis, $P' \rhd P^*$, hence $M' \rhd M^*$. □

The previous lemma immediately implies the diamond property for $\rhd$:

Lemma 4.9 (Diamond property for $\rhd$)**.** *If $M \rhd N$ and $M \rhd P$, then there exists Z such that $N \rhd Z$ and $P \rhd Z$.*

Proof. Take $Z = M^*$. □

Finally, we have a proof of the Church-Rosser Theorem:

Proof of Theorem 4.2: Since $\triangleright$ satisfies the diamond property, it follows that its reflexive transitive closure $\triangleright^*$ also satisfies the diamond property, as shown in Figure 3. But $\triangleright^*$ is the same as $\twoheadrightarrow_{\beta\eta}$ by Lemma 4.6(c), and the diamond property for $\twoheadrightarrow_{\beta\eta}$ is just the Church-Rosser property for $\to_{\beta\eta}$. □

4.5 Exercises

Exercise 12. Give a detailed proof that property (c) from Section 4.3 implies property (a).

Exercise 13. Prove that $M \triangleright M$, for all terms M.

Exercise 14. Without using Lemma 4.8, prove that $M \triangleright M^*$ for all terms M.

Exercise 15. Let $\Omega = (\lambda x.xx)(\lambda x.xx)$. Prove that $\Omega \neq_{\beta\eta} \Omega\Omega$.

Exercise 16. What changes have to be made to Section 4.4 to get a proof of the Church-Rosser Theorem for $\to_\beta$, instead of $\to_{\beta\eta}$?

Exercise 17. Recall the properties (a)–(c) of binary relations $\to$ that were discussed in Section 4.3. Consider the following similar property, which is sometimes called the "strip property":

(d)
$$\begin{tikzcd} & M \arrow[dl, two heads] \arrow[dr] & \\ N \arrow[dr, dotted, two heads] & & P \arrow[dl, dotted, two heads] \\ & Z. & \end{tikzcd}$$

Does (d) imply (a)? Does (b) imply (d)? In each case, give either a proof or a counterexample.

Exercise 18. To every lambda term M, we may associate a directed graph (with possibly multiple edges and loops) $\mathcal{G}(M)$ as follows: (i) the vertices are terms N such that $M \twoheadrightarrow_\beta N$, i.e., all the terms that M can β-reduce to; (ii) the edges are given by a single-step β-reduction. Note that the same term may have two (or

more) reductions coming from different redexes; each such reduction is a separate edge. For example, let $I = \lambda x.x$. Let $M = I(Ix)$. Then

$$\mathcal{G}(M) = \begin{tikzcd} I(Ix) \arrow[r, bend left] \arrow[r, bend right] & Ix \arrow[r] & x \end{tikzcd}.$$

Note that there are two separate edges from $I(Ix)$ to Ix. We also sometimes write bullets instead of terms, to get $\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] & \bullet \arrow[r] & \bullet \end{tikzcd}$. As another example, let $\Omega = (\lambda x.xx)(\lambda x.xx)$. Then

$$\mathcal{G}(\Omega) = \begin{tikzcd} \bullet \arrow[loop right] \end{tikzcd}.$$

(a) Let $M = (\lambda x.I(xx))(\lambda x.xx)$. Find $\mathcal{G}(M)$.

(b) For each of the following graphs, find a term M such that $\mathcal{G}(M)$ is the given graph, or explain why no such term exists. (Note: the "starting" vertex need not always be the leftmost vertex in the picture). Warning: some of these terms are tricky to find!

(i)
$$\begin{tikzcd} \bullet \arrow[r] & \bullet \arrow[loop right] \end{tikzcd}$$

(ii)
$$\begin{tikzcd} \bullet & \bullet \arrow[l] \arrow[loop right] \end{tikzcd}$$

(iii)
$$\begin{tikzcd} \bullet & \bullet \arrow[l] \arrow[r] & \bullet \end{tikzcd}$$

(iv)
$$\begin{tikzcd} \bullet & \bullet \arrow[l] \arrow[r, bend left] & \bullet \arrow[l, bend left] \end{tikzcd}$$

(v)
$$\begin{tikzcd} \bullet \arrow[loop left] & \bullet \arrow[l] \arrow[r, bend left] & \bullet \arrow[l, bend left] \arrow[r] & \bullet \arrow[loop right] \end{tikzcd}$$

(vi)
$$\begin{tikzcd} \bullet \arrow[rr] & & \bullet \arrow[dl] \\ & \bullet \arrow[ul] & \end{tikzcd}$$

(vii)
$$\begin{tikzcd} \bullet \arrow[loop left] \arrow[rr] & & \bullet \arrow[dl] \arrow[loop right] \\ & \bullet \arrow[ul] \arrow[loop below] & \end{tikzcd}$$