

**Lemma 4.6.** (a) For all  $M, M'$ , if  $M \rightarrow_{\beta\eta} M'$  then  $M \triangleright M'$ .

(b) For all  $M, M'$ , if  $M \triangleright M'$  then  $M \twoheadrightarrow_{\beta\eta} M'$ .

(c)  $\twoheadrightarrow_{\beta\eta}$  is the reflexive, transitive closure of  $\triangleright$ .

*Proof.* (a) First note that we have  $P \triangleright P$ , for any term  $P$ . This is easily shown by induction on  $P$ . We now prove the claim by induction on a derivation of  $M \rightarrow_{\beta\eta} M'$ . Please refer to pages 14 and 24 for the rules that define  $\rightarrow_{\beta\eta}$ . We make a case distinction based on the last rule used in the derivation of  $M \rightarrow_{\beta\eta} M'$ .

- If the last rule was  $(\beta)$ , then  $M = (\lambda x.Q)N$  and  $M' = Q[N/x]$ , for some  $Q$  and  $N$ . But then  $M \triangleright M'$  by (4), using the facts  $Q \triangleright Q$  and  $N \triangleright N$ .
- If the last rule was  $(\eta)$ , then  $M = \lambda x.Px$  and  $M' = P$ , for some  $P$  such that  $x \notin FV(P)$ . Then  $M \triangleright M'$  follows from (5), using  $P \triangleright P$ .
- If the last rule was  $(cong_1)$ , then  $M = PN$  and  $M' = P'N$ , for some  $P, P'$ , and  $N$  where  $P \rightarrow_{\beta\eta} P'$ . By induction hypothesis,  $P \triangleright P'$ . From this and  $N \triangleright N$ , it follows immediately that  $M \triangleright M'$  by (2).
- If the last rule was  $(cong_2)$ , we proceed similarly to the last case.
- If the last rule was  $(\xi)$ , then  $M = \lambda x.N$  and  $M' = \lambda x.N'$  for some  $N$  and  $N'$  such that  $N \rightarrow_{\beta\eta} N'$ . By induction hypothesis,  $N \triangleright N'$ , which implies  $M \triangleright M'$  by (3).

(b) We prove this by induction on a derivation of  $M \triangleright M'$ . We distinguish several cases, depending on the last rule used in the derivation.

- If the last rule was (1), then  $M = M' = x$ , and we are done because  $x \twoheadrightarrow_{\beta\eta} x$ .
- If the last rule was (2), then  $M = PN$  and  $M' = P'N'$ , for some  $P, P', N, N'$  with  $P \triangleright P'$  and  $N \triangleright N'$ . By induction hypothesis,  $P \twoheadrightarrow_{\beta\eta} P'$  and  $N \twoheadrightarrow_{\beta\eta} N'$ . Since  $\twoheadrightarrow_{\beta\eta}$  satisfies  $(cong)$ , it follows that  $PN \twoheadrightarrow_{\beta\eta} P'N'$ , hence  $M \twoheadrightarrow_{\beta\eta} M'$  as desired.
- If the last rule was (3), then  $M = \lambda x.N$  and  $M' = \lambda x.N'$ , for some  $N, N'$  with  $N \triangleright N'$ . By induction hypothesis,  $N \twoheadrightarrow_{\beta\eta} N'$ , hence  $M = \lambda x.N \rightarrow_{\beta\eta} \lambda x.N' = M'$  by  $(\xi)$ .

- If the last rule was (4), then  $M = (\lambda x.Q)N$  and  $M' = Q'[N'/x]$ , for some  $Q, Q', N, N'$  with  $Q \triangleright Q'$  and  $N \triangleright N'$ . By induction hypothesis,  $Q \twoheadrightarrow_{\beta\eta} Q'$  and  $N \twoheadrightarrow_{\beta\eta} N'$ . Therefore  $M = (\lambda x.Q)N \twoheadrightarrow_{\beta\eta} (\lambda x.Q')N' \rightarrow_{\beta\eta} Q'[N'/x] = M'$ , as desired.
- If the last rule was (5), then  $M = \lambda x.Px$  and  $M' = P'$ , for some  $P, P'$  with  $P \triangleright P'$ , and  $x \notin FV(P)$ . By induction hypothesis,  $P \twoheadrightarrow_{\beta\eta} P'$ , hence  $M = \lambda x.Px \rightarrow_{\beta\eta} P \twoheadrightarrow_{\beta\eta} P' = M'$ , as desired.

(c) This follows directly from (a) and (b). Let us write  $R^*$  for the reflexive transitive closure of a relation  $R$ . By (a), we have  $\rightarrow_{\beta\eta} \subseteq \triangleright$ , hence  $\twoheadrightarrow_{\beta\eta} = \rightarrow_{\beta\eta}^* \subseteq \triangleright^*$ . By (b), we have  $\triangleright \subseteq \twoheadrightarrow_{\beta\eta}$ , hence  $\triangleright^* \subseteq \twoheadrightarrow_{\beta\eta}^* = \twoheadrightarrow_{\beta\eta}$ . It follows that  $\triangleright^* = \twoheadrightarrow_{\beta\eta}$ .  $\square$

We will soon prove that  $\triangleright$  satisfies the diamond property. Note that together with Lemma 4.6(c), this will immediately imply that  $\twoheadrightarrow_{\beta\eta}$  satisfies the Church-Rosser property.

**Lemma 4.7** (Substitution). If  $M \triangleright M'$  and  $U \triangleright U'$ , then  $M[U/y] \triangleright M'[U'/y]$ .

*Proof.* We assume without loss of generality that any bound variables of  $M$  are different from  $y$  and from the free variables of  $U$ . The claim is now proved by induction on derivations of  $M \triangleright M'$ . We distinguish several cases, depending on the last rule used in the derivation:

- If the last rule was (1), then  $M = M' = x$ , for some variable  $x$ . If  $x = y$ , then  $M[U/y] = U \triangleright U' = M'[U'/y]$ . If  $x \neq y$ , then by (1),  $M[U/y] = y \triangleright y = M'[U'/y]$ .
- If the last rule was (2), then  $M = PN$  and  $M' = P'N'$ , for some  $P, P', N, N'$  with  $P \triangleright P'$  and  $N \triangleright N'$ . By induction hypothesis,  $P[U/y] \triangleright P'[U'/y]$  and  $N[U/y] \triangleright N'[U'/y]$ , hence by (2),  $M[U/y] = P[U/y]N[U/y] \triangleright P'[U'/y]N'[U'/y] = M'[U'/y]$ .
- If the last rule was (3), then  $M = \lambda x.N$  and  $M' = \lambda x.N'$ , for some  $N, N'$  with  $N \triangleright N'$ . By induction hypothesis,  $N[U/y] \triangleright N'[U'/y]$ , hence by (3)  $M[U/y] = \lambda x.N[U/y] \triangleright \lambda x.N'[U'/y] = M'[U'/y]$ .
- If the last rule was (4), then  $M = (\lambda x.Q)N$  and  $M' = Q'[N'/x]$ , for some  $Q, Q', N, N'$  with  $Q \triangleright Q'$  and  $N \triangleright N'$ . By induction hypothesis,  $Q[U/y] \triangleright Q'[U'/y]$  and  $N[U/y] \triangleright N'[U'/y]$ , hence by (4),  $(\lambda x.Q[U/y])N[U/y] \triangleright Q'[U'/y][N'[U'/y]/x] = Q'[N'/x][U'/y]$ . Thus  $M[U/y] = M'[U'/y]$ .

- If the last rule was (5), then  $M = \lambda x.Px$  and  $M' = P'$ , for some  $P, P'$  with  $P \triangleright P'$ , and  $x \notin FV(P)$ . By induction hypothesis,  $P[U/y] \triangleright P'[U/y]$ , hence by (5),  $M[U/y] = \lambda x.P[U/y]x \triangleright P'[U'/y] = M'[U'/y]$ .  $\square$

A more conceptual way of looking at this proof is the following: consider any derivation of  $M \triangleright M'$  from axioms (1)–(5). In this derivation, replace any axiom  $y \triangleright y$  by  $U \triangleright U'$ , and propagate the changes (i.e., replace  $y$  by  $U$  on the left-hand-side, and by  $U'$  on the right-hand-side of any  $\triangleright$ ). The result is a derivation of  $M[U/y] \triangleright M'[U'/y]$ . (The formal proof that the result of this replacement is indeed a valid derivation requires an induction, and this is the reason why the proof of the substitution lemma is so long).

Our next goal is to prove that  $\triangleright$  satisfies the diamond property. Before proving this, we first define the *maximal parallel one-step reduct*  $M^*$  of a term  $M$  as follows:

1.  $x^* = x$ , for a variable.
2.  $(PN)^* = P^*N^*$ , if  $PN$  is not a  $\beta$ -redex.
3.  $((\lambda x.Q)N)^* = Q^*[N^*/x]$ .
4.  $(\lambda x.N)^* = \lambda x.N^*$ , if  $\lambda x.N$  is not an  $\eta$ -redex.
5.  $(\lambda x.Px)^* = P^*$ , if  $x \notin FV(P)$ .

Note that  $M^*$  depends only on  $M$ . The following lemma implies the diamond property for  $\triangleright$ .

**Lemma 4.8** (Maximal parallel one-step reductions). *Whenever  $M \triangleright M'$ , then  $M' \triangleright M^*$ .*

*Proof.* By induction on the size of  $M$ . We distinguish five cases, depending on the last rule used in the derivation of  $M \triangleright M'$ . As usual, we assume that all bound variables have been renamed to avoid clashes.

- If the last rule was (1), then  $M = M' = x$ , also  $M^* = x$ , and we are done.
- If the last rule was (2), then  $M = PN$  and  $M' = P'N'$ , where  $P \triangleright P'$  and  $N \triangleright N'$ . By induction hypothesis  $P' \triangleright P^*$  and  $N' \triangleright N^*$ . Two cases:
  - If  $PN$  is not a  $\beta$ -redex, then  $M^* = P^*N^*$ . Thus  $M' = P'N' \triangleright P^*N^* = M^*$  by (2), and we are done.

- If  $PN$  is a  $\beta$ -redex, say  $P = \lambda x.Q$ , then  $M^* = Q^*[N^*/x]$ . We distinguish two subcases, depending on the last rule used in the derivation of  $P \triangleright P'$ :
  - \* If the last rule was (3), then  $P' = \lambda x.Q'$ , where  $Q \triangleright Q'$ . By induction hypothesis  $Q' \triangleright Q^*$ , and with  $N' \triangleright N^*$ , it follows that  $M' = (\lambda x.Q')N' \triangleright Q^*[N^*/x] = M^*$  by (4).
  - \* If the last rule was (5), then  $P = \lambda x.Rx$  and  $P' = R'$ , where  $x \notin FV(R)$  and  $R \triangleright R'$ . Consider the term  $Q = Rx$ . Since  $Rx \triangleright R'x$ , and  $Rx$  is a subterm of  $M$ , by induction hypothesis  $R'x \triangleright (Rx)^*$ . By the substitution lemma,  $M' = R'N' = (R'x)[N'/x] \triangleright (Rx)^*[N^*/x] = M^*$ .

- If the last rule was (3), then  $M = \lambda x.N$  and  $M' = \lambda x.N'$ , where  $N \triangleright N'$ . Two cases:
  - If  $M$  is not an  $\eta$ -redex, then  $M^* = \lambda x.N^*$ . By induction hypothesis,  $N' \triangleright N^*$ , hence  $M' \triangleright M^*$  by (3).
  - If  $M$  is an  $\eta$ -redex, then  $N = Px$ , where  $x \notin FV(P)$ . In this case,  $M^* = P^*$ . We distinguish two subcases, depending on the last rule used in the derivation of  $N \triangleright N'$ :
    - \* If the last rule was (2), then  $N' = P'x$ , where  $P \triangleright P'$ . By induction hypothesis  $P' \triangleright P^*$ . Hence  $M' = \lambda x.P'x \triangleright P^* = M^*$  by (5).
    - \* If the last rule was (4), then  $P = \lambda y.Q$  and  $N' = Q'[x/y]$ , where  $Q \triangleright Q'$ . Then  $M' = \lambda x.Q'[x/y] = \lambda y.Q'$  (note  $x \notin FV(Q')$ ). But  $P \triangleright \lambda y.Q'$ , hence by induction hypothesis,  $\lambda y.Q' \triangleright P^* = M^*$ .

- If the last rule was (4), then  $M = (\lambda x.Q)N$  and  $M' = Q'[N'/x]$ , where  $Q \triangleright Q'$  and  $N \triangleright N'$ . Then  $M^* = Q^*[N^*/x]$ , and  $M' \triangleright M^*$  by the substitution lemma.
- If the last rule was (5), then  $M = \lambda x.Px$  and  $M' = P'$ , where  $P \triangleright P'$  and  $x \notin FV(P)$ . Then  $M^* = P^*$ . By induction hypothesis,  $P' \triangleright P^*$ , hence  $M' \triangleright M^*$ .  $\square$

The previous lemma immediately implies the diamond property for  $\triangleright$ :

**Lemma 4.9** (Diamond property for  $\triangleright$ ). *If  $M \triangleright N$  and  $M \triangleright P$ , then there exists  $Z$  such that  $N \triangleright Z$  and  $P \triangleright Z$ .*

*Proof.* Take  $Z = M^*$ . □

Finally, we have a proof of the Church-Rosser Theorem:

*Proof of Theorem 4.2:* Since  $\triangleright$  satisfies the diamond property, it follows that its reflexive transitive closure  $\triangleright^*$  also satisfies the diamond property, as shown in Figure 3. But  $\triangleright^*$  is the same as  $\rightarrow_{\beta\eta}$  by Lemma 4.6(c), and the diamond property for  $\rightarrow_{\beta\eta}$  is just the Church-Rosser property for  $\rightarrow_{\beta\eta}$ . □

## 4.5 Exercises

**Exercise 12.** Give a detailed proof that property (c) from Section 4.3 implies property (a).

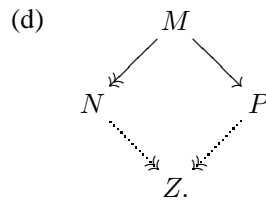
**Exercise 13.** Prove that  $M \triangleright M$ , for all terms  $M$ .

**Exercise 14.** Without using Lemma 4.8, prove that  $M \triangleright M^*$  for all terms  $M$ .

**Exercise 15.** Let  $\Omega = (\lambda x.xx)(\lambda x.xx)$ . Prove that  $\Omega \neq_{\beta\eta} \Omega\Omega$ .

**Exercise 16.** What changes have to be made to Section 4.4 to get a proof of the Church-Rosser Theorem for  $\rightarrow_{\beta}$ , instead of  $\rightarrow_{\beta\eta}$ ?

**Exercise 17.** Recall the properties (a)–(c) of binary relations  $\rightarrow$  that were discussed in Section 4.3. Consider the following similar property, which is sometimes called the “strip property”:



Does (d) imply (a)? Does (b) imply (d)? In each case, give either a proof or a counterexample.

**Exercise 18.** To every lambda term  $M$ , we may associate a directed graph (with possibly multiple edges and loops)  $\mathcal{G}(M)$  as follows: (i) the vertices are terms  $N$  such that  $M \rightarrow_{\beta} N$ , i.e., all the terms that  $M$  can  $\beta$ -reduce to; (ii) the edges are given by a single-step  $\beta$ -reduction. Note that the same term may have two (or

more) reductions coming from different redexes; each such reduction is a separate edge. For example, let  $I = \lambda x.x$ . Let  $M = I(Ix)$ . Then

$$\mathcal{G}(M) = I(Ix) \begin{array}{c} \curvearrowright \\ \longrightarrow \end{array} Ix \longrightarrow x .$$

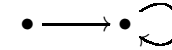
Note that there are two separate edges from  $I(Ix)$  to  $Ix$ . We also sometimes write bullets instead of terms, to get  $\bullet \begin{array}{c} \curvearrowright \\ \longrightarrow \end{array} \bullet \longrightarrow \bullet$ . As another example, let  $\Omega = (\lambda x.xx)(\lambda x.xx)$ . Then

$$\mathcal{G}(\Omega) = \bullet \begin{array}{c} \curvearrowright \\ \curvearrowright \end{array} \bullet .$$

(a) Let  $M = (\lambda x.I(xx))(\lambda x.xx)$ . Find  $\mathcal{G}(M)$ .

(b) For each of the following graphs, find a term  $M$  such that  $\mathcal{G}(M)$  is the given graph, or explain why no such term exists. (Note: the “starting” vertex need not always be the leftmost vertex in the picture). Warning: some of these terms are tricky to find!

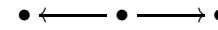
(i)



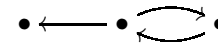
(ii)



(iii)



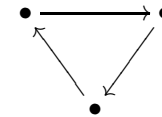
(iv)



(v)



(vi)



(vii)

