MATH/CSCI 2113, DISCRETE STRUCTURES II, Winter 2010

Handout 2: Lecture Notes on Pólya theory
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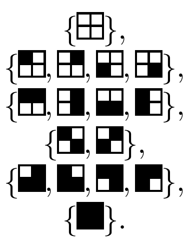
1 An example

Consider the problem of tiling a 2×2 square with the two colors black and white. Clearly, there are $2^4 = 16$ possibilities:


(1)

We now wish to know the number of different tilings up to rotation. Let us call two tilings *rotationally equivalent* if they differ only by a rotation (of $0°$, $90°$, $180°$, or $270°$). Our problem is then to count the number of equivalence classes. The equivalence classes are shown here:


(2)

We note that each equivalence classes has 1, 2, or 4 members. The more symmetries a tiling has, the smaller its equivalence class.

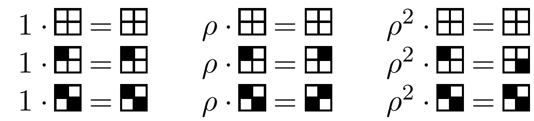
Let G be the set of available symmetry transformations. Then

$$G = \{1, \rho, \rho^2, \rho^3\}, \tag{3}$$

where 1 is the identity (rotation by $0°$), ρ is a counterclockwise rotation by $90°$, ρ^2 is a rotation by $180°$, and ρ^3 is a counterclockwise rotation by $270°$. We also call the members of G *symmetries*.

Let X be the set of 16 tilings.

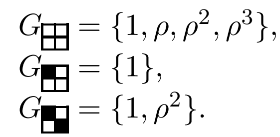
Definition. Let $g \in G$ be a symmetry, and let $x \in X$ be a tiling. Then we write $g \cdot x$ for the result of applying the transformation g to the tile x. This is called the *action* of g on X. For example:


(4)

Definition. Let $x \in X$ be a tiling, and let $g \in G$ be a symmetry. We say that g is a *symmetry for* x if $g \cdot x = x$. We write G_x for the set of symmetries for x, i.e.,

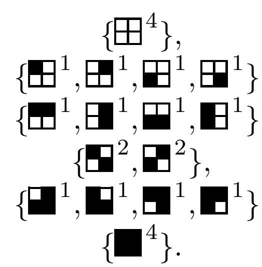
$$G_x = \{g \in G \mid g \cdot x = x\}.$$

For example,



We call $|G_x|$ the *number of symmetries* of a tiling x. Note that every tiling has at least one symmetry, namely the identity 1. However, some tilings have additional symmetries.

We now go back to our original picture of the equivalence classes. This time, we annotate each tiling with its number of symmetries.


(5)

We notice that the total number of symmetries of all the tilings in each equivalence class adds up to 4. This is the key to Pólya's counting method. We can determine the number of equivalence classes by summing the number of symmetries of all 16 tilings, then dividing the answer by 4. Since the numbers of symmetries add up to 24, the answer is $24/4 = 6$ equivalence classes.

We need a faster way to determine the total number of symmetries of all of the tilings. To clarify what we have to count, consider the following table. The rows contain symmetries, and the columns contain tilings. A checkmark ("✓") indicates that the given tiling possesses the given symmetry.

																	Total
1	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	16
ρ	✓															✓	2
ρ^2	✓									✓	✓					✓	4
ρ^3	✓															✓	2
Total	4	1	1	1	1	1	1	1	1	2	2	1	1	1	1	4	24

(6)

What we have counted is the number of checkmarks in this table. We have done this by columns, considering one tiling at a time and asking how many symmetries it has.

As faster way to count is by rows. We can consider one symmetry at a time, and ask how many tilings have this symmetry. If the number of symmetries is small, this is much faster.

To finish the example, we get that all 16 tilings are symmetric with respect to the identity 1. Only 2 tilings are symmetric with respect to each of ρ and ρ^3. And 4 of the tilings are symmetric with respect to ρ^2. Therefore, the total number of symmetries of all tilings is $16+2+4+2 = 24$. Therefore, the number of equivalence classes is

$$\frac{24}{4} = 6.$$

2 The general method

Definition. A *group* is a set G with an operation $\circ : G \times G \to G$, satisfying

(a) Associativity: $k \circ (h \circ g) = (k \circ h) \circ g$ for all $g, h, k \in G$.

(b) Unit: there exists $1 \in G$ such that for all $g \in G$, we have $1 \circ g = g = g \circ 1$. The element 1 is called the *unit* of the group.

(c) Inverse: for every $g \in G$, there exists some $h \in G$ such that $g \circ h = 1 = h \circ g$. The element h is called the *inverse* of g, and we often write $h = g^{-1}$.

One should think of G as a set of symmetry transformations, as in (3).

Definition. Let G be a group and X a set. Then an *action* of G on X is given by a function $\cdot : G \times X \to X$, satisfying:

(a) Identity: $1 \cdot x = x$,

(b) Composition: $(h \circ g) \cdot x = h \cdot (g \cdot x)$.

One should think of X as a set of objects, such as the tilings in (1). The action specifies how symmetry transformations act on objects, as we did in (4).

Definition. We say that x is a *fixed point* of g if $g \cdot x = x$. (In the above example, we said that x "has the symmetry" g). We define:

the *stabilizer* of x:	$G_x = \{g \in G \mid g \cdot x = x\}$,
the set of *fixed points* of g:	$\text{fix}(g) = \{x \in X \mid g \cdot x = x\}$,
the *orbit* of x:	$Gx = \{g \cdot x \mid g \in G\}$.

We also say that $x, y \in X$ are *equivalent*, in symbols $x \sim y$, if $y \in Gx$.

Note: in our example above, a stabilizer G_x corresponds to a column of table (6), whereas $\text{fix}(g)$ corresponds to a row. The orbit Gx corresponds to an equivalence class in (2).

Lemma 2.1. *Equivalence is a reflexive, transitive, and symmetric relation. The equivalence class of x is the orbit of x.*

Proof. Reflexivity: by the identity axiom, we have $1 \cdot x = x$, therefore $x \in Gx$, therefore $x \sim x$. Transitivity: assume $x \sim y$ and $y \sim z$. Then $y \in Gx$ and $z \in Gy$. By definition of orbits, there exist $g, h \in G$ such that $y = g \cdot x$ and $z = h \cdot y$. By the composition axiom, $z = h \cdot (g \cdot x) = (h \circ g) \cdot x$. Therefore $z \in Gx$, therefore $x \sim y$. Reflexivity: assume $x \sim y$. By definition of orbit, there exists some $g \in G$ such that $y = g \cdot x$. Then using the axioms, we get $x = 1 \cdot x = (g^{-1} \circ g) \cdot x = g^{-1} \cdot (g \cdot x) = g^{-1} \cdot y$. Therefore $x \in Gy$, hence $y \sim x$. Finally, the equivalence class of x is the set $\{y \mid x \sim y\}$, which is the orbit Gx by definition. □

Corollary 2.2. *$x \sim y$ if and only if $Gx = Gy$.* □

Our goal is to count the number of equivalence classes, i.e., the number of orbits.

Lemma 2.3 (Orbit-stabilizer theorem). *Let G be a finite group acting on a finite set X. Then for any $x \in X$,*

$$|Gx| \cdot |G_x| = |G|.$$

Proof. Fix $x \in X$. For any $y \in Gx$, we define $G_{xy} = \{g \in G \mid g \cdot x = y\}$. We first claim that G_{xy} has the same number of elements as G_x.

Indeed, let g be some group element such that $g \cdot x = y$. (This exists because $y \in Gx$). Then we can define functions $\varphi : G_x \to G_{xy}$ and $\psi : G_{xy} \to G_x$ by $\varphi(h) = g \circ h$ and $\psi(h) = g^{-1} \circ h$. Note that for all $h \in G_x$, we have

$$\varphi(h) \cdot x = (g \circ h) \cdot x = g \cdot (h \cdot x) = g \cdot x = y,$$

and therefore $\varphi(h)$ is indeed an element of G_{xy}. Conversely, for any $h \in G_{xy}$, we have

$$\begin{aligned}\psi(h) \cdot x &= (g^{-1} \circ h) \cdot x = g^{-1} \cdot (h \cdot x) = g^{-1} \cdot y = g^{-1} \cdot (g \cdot x) \\ &= (g^{-1} \circ g) \cdot x = 1 \cdot x = x,\end{aligned}$$

and therefore $\psi(h)$ is indeed an element of G_x. Therefore, φ and ψ are well-defined functions. They are also each other's inverses, and therefore they establish a bijection between G_x and G_{xy}.

Now let $Gx = \{y_1, \ldots, y_n\}$. (Note that, because x is a member of its own orbit, one of the y_i's must be equal to x itself). We claim that the sets $G_{xy_1}, \ldots, G_{xy_n}$ are disjoint, and that their union is G.

To prove disjointness, assume that $G_{xy_i} \cap G_{xy_j}$ is non-empty for some $i \neq j$. Then there exists some $g \in G_{xy_i} \cap G_{xy_j}$. By definition of G_{xy_i}, it follows that $g \cdot x = y_i$, and similarly $g \cdot x = y_j$. Therefore we have $y_i = y_j$, contradicting $i \neq j$. To prove that $G = G_{xy_1} \cup \ldots \cup G_{xy_n}$, let some arbitrary $g \in G$ be given. Then $g \cdot x \in Gx$, therefore $g \cdot x = y_i$ for some i. It follows that $g \in G_{xy_i}$.

We now finish the proof of the lemma: because G is a disjoint union of $G_{xy_1}, \ldots, G_{xy_n}$, we have

$$|G| = |G_{xy_1}| + \ldots + |G_{xy_n}|.$$

Moreover, since $|G_{xy_i}| = |G_x|$ for all i, we have

$$|G| = n|G_x|.$$

But $n = |Gx|$, and so we have $|G| = |Gx| \cdot |G_x|$ as desired. □

The orbit-stabilizer theorem corresponds to our observation, in (5), that the number of symmetries on each tiling x (i.e., $|G_x|$), times the size of its equivalence class (i.e., $|Gx|$), is equal to 4 (i.e., $|G|$).

Corollary 2.4. *If H is any orbit, then $\sum_{x \in H} |G_x| = |G|$.*

Proof. First, note that if $x \sim y$, then $|G_x| = |G_y|$. This is a consequence of Corollary 2.2 and Lemma 2.3. Now suppose $H = Gx = \{y_1, \ldots, y_n\}$. Then

$$\sum_{x \in H} |G_x| = |G_{y_1}| + \ldots + |G_{y_n}| = n|G_x| = |Gx| \cdot |G_x| = |G|.$$

Lemma 2.5 (Burnside's lemma). *Let G be a finite group acting on a finite set X. Then the number of orbits is equal to*

$$\frac{1}{|G|} \sum_{g \in G} |\operatorname{fix}(g)|. \tag{7}$$

Proof. The proof is inspired by table (6). As we have already remarked, $|\operatorname{fix}(g)|$ is the number of checkmarks in row g of that table, whereas $|G_x|$ is the number of checkmarks in column x. More formally, define the function

$$f(g,x) = \begin{cases} 1 & \text{if } g \cdot x = x, \\ 0 & \text{else.} \end{cases}$$

Then

$$|\operatorname{fix}(g)| = \sum_{x \in X} f(g,x) \text{ and} \tag{8}$$

$$|G_x| = \sum_{g \in G} f(g,x). \tag{9}$$

The remainder of the proof is a calculation. Let Orb be the set of orbits.

$$\begin{aligned} \textstyle\sum_{g\in G} |\operatorname{fix}(g)| &= \textstyle\sum_{g\in G}\sum_{x\in X} f(g,x) && \text{by (8)} \\ &= \textstyle\sum_{x\in X}\sum_{g\in G} f(g,x) && \text{reverse order of sums} \\ &= \textstyle\sum_{x\in X} |G_x| && \text{by (9)} \\ &= \textstyle\sum_{H\in \text{Orb}}\sum_{x\in H} |G_x| && \text{rearrange sum} \\ &= \textstyle\sum_{H\in \text{Orb}} |G| && \text{by Corollary 2.4} \\ &= |\text{Orb}| \cdot |G|. \end{aligned}$$

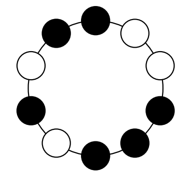
The lemma follows by dividing by $|G|$. □

Note that Burnside's lemma can be summarized as saying that the number of orbits of a group action is equal to the average number of fixed points of the group elements.

Pólya's counting method is the application of Burnside's lemma in the case where G is a group of symmetries on some number of slots that are to be filled with objects (such as colors). The Pólya enumeration theorem is actually a generalization of Burnside's lemma that applies to more general situations, such as when the objects are weighted. However, we will not need those generalizations here.

3 More examples

Problem 1. Consider circular necklaces made up of 10 colored beads. If there are 2 colors available, then there are $2^{10} = 1024$ such necklaces. Here is one example:



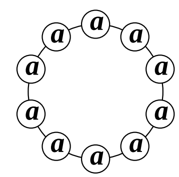
We call two necklaces equivalent if they differ by a rotation. How many equivalence classes are there?

The symmetry group in question is

$$G = \{1, \rho, \rho^2, \ldots, \rho^9\},$$

where ρ is a counterclockwise rotation by 1 bead. Clearly, $\rho^{10} = 1$. We apply Burnside's lemma to count the number of orbits.

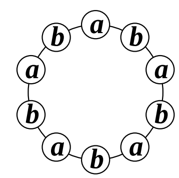
- $\operatorname{fix}(1)$ consists of all 1024 necklaces.
- Each of $\operatorname{fix}(\rho)$, $\operatorname{fix}(\rho^3)$, $\operatorname{fix}(\rho^7)$, and $\operatorname{fix}(\rho^9)$ consists of necklaces of the form



(10)

  where a is some color (either black or white). So there are 2 fixed points in each case.

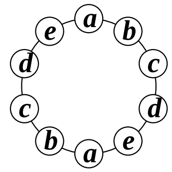
- Each of $\operatorname{fix}(\rho^2)$, $\operatorname{fix}(\rho^4)$, $\operatorname{fix}(\rho^6)$, $\operatorname{fix}(\rho^8)$ consists of all necklaces of the form



(11)

where a, b are arbitrary colors. So there are $2^2 = 4$ fixed points in these cases.

- $\text{fix}(\rho^5)$ consists of all necklaces of the form



(12)

where a, b, c, d, e are arbitrary colors. So ρ^5 has $2^5 = 32$ fixed points.

By the Burnside lemma, the number of orbits is

$$\frac{1}{10}(1024 + 2 + 4 + 2 + 4 + 32 + 4 + 2 + 4 + 2) = \frac{1080}{10} = 108.$$

Problem 2. Consider circular necklaces of 10 beads as in Problem 1, but with n colors available instead of 2.

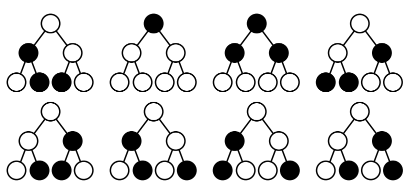
The reasoning is exactly the same. The number of necklaces is n^{10}. All of them are fixed points for 1. The other fixed points are again of the form (10), (11), or (12), and there are respectively n, n^2, and n^5 of them. By the Burnside lemma, the number of orbits is

$$\frac{1}{10}(n^{10} + 4n + 4n^2 + n^5).$$

It is interesting that in all Pólya-type problems, if n is the number of colors, the answer is always a polynomial in n.

Problem 3. How many circular necklaces can one make from 10 black or white beads, if rotations and reflections are taken into account?

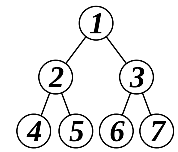
Problem 4. Consider rooted binary trees of depth 2, with vertices colored black and white. Here are some examples:



(13)

We want to consider trees up to a re-ordering of branches, i.e., where left and right don't matter. For example, all the trees in the first row of (13) are non-equivalent, but all the trees in the second row are equivalent to each other. How many equivalence classes of trees are there?

For convenience, we number the locations as follows.



There are 2^7 colorings if symmetries are not taken into account.

There are three basic symmetries: α, which swaps the subtree at 2 with the subtree at 3; β, which swaps 4 and 5, and γ, which swaps 6 and 7. Written as permutations (in cycle notation), we have

$$\alpha = (1)(23)(46)(57), \beta = (1)(2)(3)(45)(6)(7), \gamma = (1)(2)(3)(4)(5)(67)$$

Note that $\gamma \circ \alpha = \alpha \circ \beta$, $\beta \circ \alpha = \alpha \circ \gamma$, and $\gamma \circ \beta = \beta \circ \gamma$. Also, we have $\alpha^2 = \beta^2 = \gamma^2 = 1$. Altogether, the symmetry group consists of 8 elements (we drop the symbol "$\circ$" when convenient):

$$G = \{1, \alpha, \beta, \alpha\beta, \gamma, \alpha\gamma, \beta\gamma, \alpha\beta\gamma\}.$$

We can write each group element as a permutation of vertices:

$$\begin{array}{lcl} 1 & = & (1)(2)(3)(4)(5)(6)(7) \\ \alpha & = & (1)(23)(46)(57) \\ \beta & = & (1)(2)(3)(45)(6)(7) \\ \alpha\beta & = & (1)(23)(5647) \\ \gamma & = & (1)(2)(3)(4)(5)(67) \\ \alpha\gamma & = & (1)(23)(6574) \\ \beta\gamma & = & (1)(2)(3)(45)(67) \\ \alpha\beta\gamma & = & (1)(23)(47)(56) \end{array} \qquad (14)$$

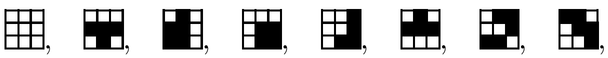
We need to determine the number of fixed points for each group element. They are related to the number of cycles in each permutation in (14).

Namely, for a certain coloring to be a fixed point for g, all the vertices that are part of a cycle of g must receive the same color. Therefore, the number of fixed points of each group element g is 2^c, where c is the number of cycles in the corresponding permutation in (14). The numbers of fixed points of the respective group elements are 2^7, 2^4, 2^6, 2^3, 2^6, 2^3, 2^5, 2^4.

Therefore by Burnside's lemma, the number of orbits is

$$\frac{1}{8}(2^7 + 2^4 + 2^6 + 2^3 + 2^6 + 2^3 + 2^5 + 2^4) = 42.$$

Problem 5. Consider tilings of a 3×3 square with two colors, such as the following:



How many different tilings are there (a) up to rotation, and (b) up to rotation and mirror images?

Problem 6. (a) In how many ways can the faces of a cube be colored with two colors, up to a rotation of the cube? Hint: there are 24 rotations of the cube, including the identity. (b) Answer the same question for n colors. (c) Answer the same question for rotations and mirror images. (d) Answer the same question for an octahedron.

Problem 7. (a) How many circular necklaces of length 11 can you make, up to rotation, from black and white beads? (b) How many of length 12?

Problem 8. Among the tilings from Problem 5, consider those that consist of exactly 5 black tiles and 4 white tiles. Up to rotation, how many such tilings are there?