Math 4680, Topics in Logic and Computation, Winter 2012

Answers to Homework 1

Problem 1.1 #2 Let $P(\alpha)$ be the property "the length of α is not 0, 2, 3, or 6.". We prove that any well-formed formula α has this property by induction on well-formed formulas. If $\alpha = \mathbf{A}_n$ is atomic, then it has length 1, and thus it satisfies the property P. The same is true if $\alpha = \top$ or $\alpha = \bot$. Now suppose that $\alpha = (\neg\beta)$ for some well-formed formula β. If n is the length of β, then the length of α is $n+3$. By induction hypothesis, we know that n is not -3, -1, 0, or 3. Thus, $n+3$ is not 0, 2, 3, or 6 and α has the property P. Finally, suppose $\alpha = (\beta \square \gamma)$ for some well-formed formulas β and γ, where $\square \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$. Let n and m be the lengths of β and γ, respectively. Then α has length $n+m+3$. Either n and m are both 1, in which case $n+m+3=5$. Or otherwise, at least one of n and m is $\geqslant 4$, in which case $n+m+3 \geqslant 8$. In any case, $n+m+3$ is not 0, 2, 3, or 6, and thus α has the property P. This proves the claim.

To show that all other lengths are possible, we only need to give a well-formed formula for each length.

$\mathbf{A}_1$	has length 1.
$(\neg\mathbf{A}_1)$	has length 4.
$(\mathbf{A}_1 \wedge \mathbf{A}_2)$	has length 5.
$(\neg(\neg\mathbf{A}_1))$	has length 7.
$(\neg(\mathbf{A}_1 \wedge \mathbf{A}_2))$	has length 8.
$((\mathbf{A}_1 \wedge \mathbf{A}_2) \wedge \mathbf{A}_3)$	has length 9.

After this, we can generate any length because if α has length n, then $(\neg\alpha)$ has length $n+3$.

Problem 1.1 #3 If α is a well-formed formula, we write $c(\alpha)$ for the number of occurrences of binary connectives in α, and $s(\alpha)$ for the number of occurrences of sentence symbols and the symbols $\top$ and $\bot$ in α. We claim that $s(\alpha) = c(\alpha)+1$. Proof by induction on well-formed formulas: If $\alpha = \mathbf{A}_n$ is a sentence symbol, or if α is $\top$ or $\bot$, then $s(\alpha)=1$ and $c(\alpha)=0$, thus the claim holds. If $\alpha = (\neg\beta)$ for a well-formed formula β, then $s(\alpha)=s(\beta)$ and $c(\alpha)=c(\beta)$, so the claim holds by induction hypothesis. Finally, if $\alpha = (\beta \square \gamma)$, where $\square \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$, then $s(\alpha) = s(\beta)+s(\gamma)$, and $c(\alpha) = c(\beta)+c(\gamma)+1$. By induction hypothesis, we have

$$s(\alpha) = s(\beta)+s(\gamma) = (c(\beta)+1)+(c(\gamma)+1) = c(\alpha)+1,$$

which proves the claim.

Problem 1.2 #2 (a) Yes, $(((P \to Q) \to P) \to P)$ is a tautology, as the following truth table shows:

P	Q	$P \to Q$	$(P \to Q) \to P$	$((P \to Q) \to P) \to P$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	F	T

Moreover, from the truth table, we see that $((P \to Q) \to P)$ is tautologically equivalent to P.

(b) Let $\sigma_0 = (P \to Q)$ and $\sigma_{k+1} = (\sigma_k \to P)$. By induction we prove that for all $k \geqslant 1$: if k is odd, then $\sigma_k \models\!\!\!\!\!\;\Dashv P$, and if k is even, then $\models \sigma_k$. Proof: Base case: for $k=1$, this is seen from the above truth table. Indunction step: If k is even, then $k+1$ is odd. By induction hypothesis, σ_k is a tautology, so $\sigma_{k+1} = (\sigma_k \to P)$ is logically equivalent to P. On the other hand, if k is odd, then $k+1$ is even. By induction hypothesis, σ_k is logically equivalent to P, so $\sigma_{k+1} = (\sigma_k \to P)$ is logically equivalent to $P \to P$, therefore a tautology.

Problem 1.2 #5 (a) The assertion is true. Proof: Assume $\Sigma \models \alpha$. We have to show $\Sigma \models \alpha \vee \beta$. So let v be any truth assignment satisfying all the formulas in Σ. By hypothesis, $\bar{v}(\alpha) = T$. By definition of $\bar{v}$, we have $\bar{v}(\alpha \vee \beta) = T$. Since v was arbitrary, we have $\Sigma \models \alpha \vee \beta$ as desired. The argument is similar if $\Sigma \models \beta$ is first assumed.

(b) The assertion is false. Counterexample: Let $\Sigma = \emptyset$, $\alpha = A$, and $\beta = \neg A$, where A is a sentence symbol. Then $\Sigma \models \alpha \vee \beta$, but $\Sigma \not\models \alpha$ and $\Sigma \not\models \beta$.

Another counterexample is: Let $\alpha = A$, $\beta = B$, and $\Sigma = A \vee B$, where A, B are two different sentence symbols.

Problem 1.2 #7 You can ask the person: "If I asked you whether I should take the right path, would you answer 'yes?' "

Problem 1.2 #10 We start with a lemma that will be useful in this problem and elsewhere.

Lemma. *Two sets of wffs Σ and Σ' are equivalent if and only if $\Sigma \models \alpha'$ for all $\alpha' \in \Sigma'$ and $\Sigma' \models \alpha$ for all $\alpha \in \Sigma$.*

Proof. The left-to-right implication is trivial; if Σ and Σ' are equivalent, then from $\alpha \in \Sigma$ we trivially get $\Sigma \models \alpha$, hence $\Sigma' \models \alpha$, and similarly for the second part. The right-to-left implication is also easy: Assume $\Sigma \models \alpha'$ for all $\alpha' \in \Sigma'$ and $\Sigma' \models \alpha$ for all $\alpha \in \Sigma$. Let v be any truth assignment satisfying Σ; then v also satisfies Σ' (because every $\alpha' \in \Sigma'$ is a consequence of Σ); conversely, every truth assignment satisfying Σ' also satisfies Σ. It follows that Σ and Σ' have the same tautological consequences. $\square$

(a) We claim that any finite set Σ of wffs has an independent equivalent subset. Proof: by induction on the number of elements of Σ. Base case: If Σ is empty, then it is independent, so there is nothing to show. Induction step: Consider a set Σ of size $n+1$. Case 1: if Σ is independent, there is nothing to show. Case 2: if Σ is not independent, then there exists some $\alpha \in \Sigma$ such that $\Sigma - \{\alpha\} \models \alpha$. By the Lemma, Σ and $\Sigma - \{\alpha\}$ are equivalent. By induction hypothesis, $\Sigma - \{\alpha\}$ has some independent equivalent subset Σ', which is then also an independent equivalent subset of Σ.

(b) Let $\alpha_k = A_0 \wedge \ldots \wedge A_k$, for all $k \geqslant 0$, and let $\Sigma = \{\alpha_k \mid k \in \mathbb{N}\}$. We claim that Σ has no independent equivalent subset. Indeed, let $\Gamma \subseteq \Sigma$ be a subset. There are three cases to consider: Case 1: Γ is empty. In this case $\Gamma \not\models \alpha_0$, so Γ is not equivalent to Σ. Case 2: Γ has exactly one element, say α_k. In this case, $\Gamma \not\models \alpha_{k+1}$, hence Γ is not equivalent to Σ. Case 3: Γ has at least two elements, say α_k and α_m, where $k < m$. Then $\alpha_m \models \alpha_k$, so Γ is not independent. In all three cases, Γ fails to be an independent equivalent subset of Σ, so Σ has no such subset.

(c) Let $\Sigma = \{\sigma_0, \sigma_1, \ldots\}$. Define $\tau_0 = \sigma_0$, $\tau_1 = \sigma_0 \rightarrow \sigma_1$, $\tau_2 = \sigma_0 \wedge \sigma_1 \rightarrow \sigma_2$, and so on; in general, for $k \geqslant 0$:

$$\tau_k = \sigma_0 \wedge \ldots \wedge \sigma_{k-1} \rightarrow \sigma_k.$$

Consider the set $\Gamma = \{\tau_0, \tau_1, \ldots\}$. We claim that Γ is equivalent to Σ. Indeed, on the one hand, we have $\sigma_k \models \tau_k$, therefore $\Sigma \models \tau_k$ for all k; on the other hand, we can prove $\tau_0, \ldots, \tau_k \models \sigma_k$ by an easy induction, hence $\Gamma \models \sigma_k$ for all k, so by the Lemma, Γ and Σ are equivalent.

In general, Γ is not independent. Indeed, certain elements $\tau_k \in \Gamma$ may be tautologies. Now consider the subset

$$\Gamma' = \{\tau_k \in \Gamma \mid \tau_k \text{ is not a tautology}\}.$$

Then clearly (using the Lemma), Γ' is equivalent to Γ. Moreover, we claim that Γ' is independent. Indeed, let $\tau_k \in \Gamma'$. Since τ_k is not a tautology, there exists some truth assignment v such that $\bar{v}(\tau_k) = F$. Since $\tau_k = \sigma_0 \wedge \ldots \wedge \sigma_{k-1} \rightarrow \sigma_k$, it follows that $\bar{v}(\sigma_i) = T$ for all $i < k$, and $\bar{v}(\sigma_k) = F$. Then for all $i < k$, we have $\sigma_i \models \tau_i$, hence $\bar{v}(\tau_i) = T$. And for all $j > k$, we have $\bar{v}(\sigma_0 \wedge \ldots \wedge \sigma_{j-1}) = F$, hence $\bar{v}(\tau_j) = T$. Therefore the truth assignment v makes τ_k false, and all other formulas of Γ' true. It follows that τ_k is not tautologically implied by the other formulas of Γ', hence Γ' is independent as desired.

Problem 1.4 #2 The string $(\mathbf{A}_3 \rightarrow \wedge \mathbf{A}_4)$ has length 6. Therefore it is not a well-formed formula by Problem 1.1 #2.