Math 4680, Topics in Logic and Computation, Winter 2012

Answers to Homework 2

Problem 1.5 #1 (a) $G(x, y, z) = (\neg x \wedge \neg y \wedge \neg z) \vee (\neg x \wedge \neg y \wedge z) \vee (\neg x \wedge y \wedge \neg z) \vee (x \wedge \neg y \wedge \neg z)$

(b) $G(x, y, z) = (y \vee z) \to (\neg(x \vee (y \wedge z))$

Problem 1.5 #4 (a) To show that $\{M, \bot\}$ is complete, it suffices to show that the Boolean formulas $A \mapsto \neg A$ and $(A, B) \mapsto A \wedge B$ can be expressed. We have:

$$\begin{aligned} &\neg A = M(A, A, A) \\ &A \wedge B = \neg M(A, B, \bot) = M(M(A, B, \bot), M(A, B, \bot), M(A, B, \bot)). \end{aligned}$$

(b) To show that $\{M\}$ is not complete, consider any formula φ constructed from Boolean variables x and y by (possibly repeated) application of M. We prove by induction: φ is logically equivalent to either x or y or $\neg x$ or $\neg y$.

Base case: if φ is a variable, then φ is either x or y, so the claim trivially holds.

Induction step: suppose $\varphi = M(\varphi_1, \varphi_2, \varphi_3)$. By induction hypothesis, each of φ_1, φ_2, and φ_3 is logically equivalent to one of x, y, $\neg x$, or $\neg y$. Then we must have either $\varphi_i \models\!\dashv \varphi_j$ for some $i \neq j$, or $\varphi_i \models\!\dashv \neg \varphi_j$ for some $i \neq j$.

Case 1: $\varphi_i \models\!\dashv \varphi_j$ for some $i \neq j$. Without loss of generality, $\varphi_1 \models\!\dashv \varphi_2$. In this case:

$$\varphi = M(\varphi_1, \varphi_2, \varphi_3) \models\!\dashv M(\varphi_1, \varphi_1, \varphi_3) \models\!\dashv \neg \varphi_1.$$

Case 2: $\varphi_i \models\!\dashv \neg \varphi_j$ for some $i \neq j$. Without loss of generality, $\varphi_1 \models\!\dashv \neg \varphi_2$. In this case:

$$\varphi = M(\varphi_1, \varphi_2, \varphi_3) \models\!\dashv M(\varphi_1, \neg \varphi_1, \varphi_3) \models\!\dashv \neg \varphi_3.$$

In either case, the claim follows by induction hypothesis.

Finally, you may wonder whether one can perhaps construct a formula $\varphi(x_1, x_2, \ldots, x_n)$ using *more* than 2 variables, such that $\varphi(x_1, x_2, \ldots, x_n)$ is logically equivalent to $x_1 \wedge x_2$. However, this is clearly not the case, because then $\varphi(x_1, x_2, x_2, \ldots, x_2)$ is also logically equivalent to $x_1 \wedge x_2$, and it uses only 2 variables, so the above argument applies to it.

Problem 1.7 #12 (a) $\{A, \neg A\}$.

(b) $\{A, B, \neg(A \wedge B)\}$.

(c) $\{A, B, C, \neg(A \wedge B \wedge C)\}$.

Problem 2.1 #1 Recall the restricted quantifiers:

- For all numbers x, ...: $\forall x(N(x) \to (...))$.
- There is a number x such that ...: $\exists x(N(x) \wedge (...))$.
- There is no number x such that ...: $\neg \exists x(N(x) \wedge (...))$.

Translations:

(a) $\forall x(N(x) \to 0 < x)$.

(b) $\forall x(N(x) \to I(x) \to I(0))$ or equivalently $(\exists x(N(x) \wedge I(x))) \to I(0)$.

(c) $\neg \exists x.N(x) \wedge x < 0$.

(d) $\forall x.[(N(x) \wedge \neg I(x) \wedge (\forall y.[(N(y) \wedge y < x) \to I(y)])) \to I(x)]$.

(e) $\neg \exists x.[N(x) \wedge \forall y.(N(y) \to y < x)]$.

(f) $\neg \exists x.[N(x) \wedge \neg \exists y.[N(y) \wedge y < x]]$.

Problem 2.2 #2 (a) Consider the structure $\mathfrak{A}$ with $|\mathfrak{A}| = \{a, b, c\}$ and $P = \{(a, b), (b, c)\}$. This satisfies (b) and (c), but not (a).

(b) Consider the structure $\mathfrak{B}$ with $|\mathfrak{B}| = \{a, b\}$ and with the predicate $P = \{(a, a), (a, b), (b, a), (b, b)\}$. This satisfies (a) and (c) but not (b).

(c) Consider the structure $\mathfrak{C}$ with $|\mathfrak{C}| = \{a, b\}$ and with $P = \{(a, a), (b, b)\}$. This satisfies (a) and (b) but not (c).

Problem 2.2 #8 "$\Rightarrow$": We prove the contrapositive. Assume $\Sigma \not\models \tau$. By assumption, we have $\Sigma \models \neg\tau$. Since $\mathfrak{A}$ is a model of Σ, it follows by definition of logical consequence that $\models_{\mathfrak{A}} \neg\tau$, hence $\not\models_{\mathfrak{A}} \tau$, as desired.

"$\Leftarrow$": Assume $\Sigma \models \tau$. Since $\mathfrak{A}$ is a model of Σ, it follows by definition of logical consequence that $\models_{\mathfrak{A}} \tau$, as desired.

Problem 2.2 #11 For greater clarity, we write "$\equiv$" for equality in the metalanguage and "$=$" for equality in the object language.

(a) $\varphi_a(x) \equiv \forall y(x + y = y)$.

(b) $\varphi_b(x) \equiv \forall y(x \cdot y = y)$.

(c) $\varphi_c(x, y) \equiv \exists z(\varphi_b(z) \wedge x + z = y)$.

(d) $\varphi_d(x, y) \equiv \neg x = y \wedge \exists z(x + z = y)$.

Problem 2.2 #15 Let $p_1, p_2, p_3, \ldots = 2, 3, 5, 7, \ldots$ be the list of all prime numbers. Recall that every natural number $n > 0$ has a unique factorization into primes: $n = p_1^{k_1} \cdot p_2^{k_2} \cdot p_3^{k_3} \cdot \ldots$, where all but finitely many of $k_1, k_2, k_3, \ldots$ are 0. Define the following function $f : \mathbb{N} \to \mathbb{N}$:

$$\begin{aligned} f(0) &= 0, \\ f(p_1^{k_1} \cdot p_2^{k_2} \cdot p_3^{k_3} \cdot \ldots) &= p_1^{k_2} \cdot p_2^{k_1} \cdot p_3^{k_3} \cdot \ldots. \end{aligned}$$

Note how k_1 and k_2 have been swapped on the right-hand side. Then it is easy to see that for all $n, m \in \mathbb{N}$, $f(n \cdot m) = f(n) \cdot f(m)$. Indeed, if n or m is 0, then this is a triviality. If they are both non-zero, they have prime factorizations $n = p_1^{k_1} \cdot p_2^{k_2} \cdot p_3^{k_3} \cdot \ldots$ and $m = p_1^{l_1} \cdot p_2^{l_2} \cdot p_3^{l_3} \cdot \ldots$, and we have

$$\begin{aligned} f(nm) &= f(p_1^{k_1+l_1} \cdot p_2^{k_2+l_2} \cdot p_3^{k_3+l_3} \cdot \ldots) \\ &= p_1^{k_2+l_2} \cdot p_2^{k_1+l_1} \cdot p_3^{k_3+l_3} \cdot \ldots \\ &= f(n)f(m). \end{aligned}$$

If follows that $f : \mathbb{N} \to \mathbb{N}$ is an automorphism of $(\mathbb{N}; \cdot)$. By the homomorphism theorem, it follows that any formula φ satisfies

$$\models_{\mathbb{N}} \varphi[s] \Longleftrightarrow \models_{\mathbb{N}} \varphi[f \circ s]. \tag{1}$$

Suppose now that $\varphi(x, y, z)$ were a formula defining addition, i.e.,

$$\models_{\mathbb{N}} \varphi(x, y, z)[s] \Longleftrightarrow s(x) + s(y) = s(z). \tag{2}$$

Putting (1) and (2) together, we have

$$s(x) + s(y) = s(z) \Longleftrightarrow f(s(x)) + f(s(y)) = f(s(z)). \tag{3}$$

Now choose s so that $s(x) = 2$, $s(y) = 5$, and $s(z) = 7$. From (3), we have

$$2 + 5 = 7 \Longleftrightarrow f(2) + f(5) = f(7). \tag{4}$$

However, $f(2) = 3$, $f(5) = 5$, and $f(7) = 7$, so the right-hand side is false whereas the left-hand side is true. This is a contradiction; hence addition is not definable by any formula $\varphi(x, y, z)$ in the language with only multiplication.