Math 4680, Topics in Logic and Computation, Winter 2012

Midterm answers, March 2, 2012

Problem 1 (Source: Enderton, 1.2 #13)**.** An advertisement for a tennis magazine states: "If I am not playing tennis, I am watching tennis. And if I am not watching tennis, I am reading about tennis". (a) Translate this into propositional logic. (b) Assuming the speaker cannot do more that one of these activities at a time, then what is the speaker doing?

Answer: (a) Using P for "I am playing tennis", W for "I am watching tennis", and R for "I am reading tennis", the translation is

$$(\neg P \to W) \wedge (\neg W \to R).$$

(b) We examine the relevant rows of the truth table:

P	W	R	$\neg P \to W$	$\neg W \to R$	$(\neg P \to W) \wedge (\neg W \to R)$
T	F	F	T	F	F
F	T	F	T	T	T
F	F	T	F	T	F

So the speaker must be watching tennis.

Problem 2. Let C be the ternary *consensus* connective: $\mathsf{C}(\alpha, \beta, \gamma)$ is true if α, β, and γ agree, and otherwise false. In other words, C is defined by the following truth table:

α	β	γ	$\mathsf{C}(\alpha, \beta, \gamma)$
T	T	T	T
T	T	F	F
T	F	T	F
T	F	F	F
F	T	T	F
F	T	F	F
F	F	T	F
F	F	F	T

(a) Show that $\alpha \wedge \beta$, $\alpha \vee \beta$, and $\alpha \to \beta$ are all definable in terms of C.

Answer: We can define, in this order,

$$\begin{aligned}
\top &= \mathsf{C}(\alpha, \alpha, \alpha), \\
\alpha \wedge \beta &= \mathsf{C}(\alpha, \beta, \top), \\
\alpha \to \beta &= \mathsf{C}(\alpha, \alpha, \alpha \wedge \beta), \\
\alpha \leftrightarrow \beta &= \mathsf{C}(\alpha, \alpha, \beta), \\
\alpha \vee \beta &= \mathsf{C}(\alpha \wedge \beta, \alpha \wedge \beta, \alpha \leftrightarrow \beta).
\end{aligned}$$

(b) Prove that $\{\mathsf{C}, \bot\}$ is complete.

Answer: We have $\neg \alpha \models\!\!\!\dashv \alpha \to \bot$. Thus, using part (a), $\neg$ and $\wedge$ are definable from C and $\bot$. But $\{\wedge, \neg\}$ is a complete set of connectives.

(c) Prove that $\{\mathsf{C}\}$ is not complete. Hint: consider a truth assignment that assigns T to every sentence symbol.

Answer: Let v_0 be the truth assignment that assigns T to every sentence symbol. Every formula α built from sentence symbols and C has the property that $\bar{v}_0(\alpha) = T$. Proof: by induction on α. If $\alpha = \mathbf{A}_n$ is a sentence symbol, then $\bar{v}_0(\alpha) = v_0(\mathbf{A}_n) = T$. If $\alpha = \mathsf{C}(\alpha_1, \alpha_2, \alpha_3)$, then $\bar{v}_0(\alpha_i) = T$, for $i = 1, 2, 3$, by induction hypothesis. Thus $\bar{v}_0(\alpha) = \mathsf{C}(T, T, T) = T$. This concludes the induction. Notice that $v_0(\bot) = F$, thus $\bot$ is not definable in terms of C. $\square$

Problem 3. Find an unsatisfiable set of 4 formulas, such that every 3-element subset is satisfiable.

Answer: A possible such set is $\{A, B, C, \neg(A \wedge B \wedge C)\}$.

Problem 4. Show that neither of the following sentences logically implies the other. In each case, do this by giving a structure in which one sentence is true and the other false.

(a) $\forall x \exists y \neg P(x, y)$

(b) $\neg \forall x \exists y P(x, y)$

Answer: Consider models $\mathfrak{A}$ and $\mathfrak{B}$ with $|\mathfrak{A}| = |\mathfrak{B}| = \{a, b\}$ and $P_{\mathfrak{A}} = \{(a, a), (b, b)\}$ and $P_{\mathfrak{B}} = \{(a, a), (a, b)\}$. Then:

$$\begin{array}{ll} \models_{\mathfrak{A}} (a) & \not\models_{\mathfrak{B}} (a) \\ \not\models_{\mathfrak{A}} (b) & \models_{\mathfrak{B}} (b) \end{array}$$

Therefore $\mathfrak{A}$ demonstrates that $(a) \not\models (b)$, and $\mathfrak{B}$ demonstrates that $(b) \not\models (a)$.

Problem 5. For each of the following relations, give a formula which defines it in $(\mathbb{N}; +, \cdot)$. The language is assumed to have equality.

(a) $\{\langle m, n\rangle \mid n$ is the successor of m in $\mathbb{N}\}$.

Answer: $\varphi(x, y) \equiv \exists z(x + z = y \wedge \forall w(z \cdot w = w))$.

(b) $\{\langle m, n\rangle \mid m < n$ in $\mathbb{N}\}$.

Answer: $\varphi(x, y) \equiv \exists z(x + z = y) \wedge x \neq y$.

(c) $\{\langle m, n\rangle \mid$ the greatest common divisor of n and m is $1\}$.

Answer: $\varphi(x, y) \equiv \forall d(\exists k(d \cdot k = x) \wedge \exists k'(d \cdot k' = y) \rightarrow d = 1)$.