MATH 2135, LINEAR ALGEBRA, Winter 2013

Handout 6: Problems on inner product spaces
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Problem 1. Find an orthonormal basis of $\mathbb{R}^3$ containing the vector $\frac{1}{3}(1,2,2)$ as the first basis vector.

Problem 2. Consider the subspace of $\mathbb{R}^4$ spanned by $(1,1,0,0)$, $(1,0,1,0)$, and $(1,0,0,1)$. Use the Gram-Schmidt method to find an orthogonal basis of this subspace.

Problem 3. On $\mathbb{R}^3$, consider the inner product defined by $\langle v,w\rangle = v^T Aw$, where

$$A = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 3 \end{pmatrix}$$

Use the Gram-Schmidt method to find a basis of $\mathbb{R}^3$ that is orthonormal with respect to this inner product.

Problem 4. Consider the vector space $V = C[0,1]$ of continuous, real-valued functions defined on the unit interval $[0,1] = \{x \in \mathbb{R} \mid 0 \leqslant x \leqslant 1\}$. Consider the inner product on V that is defined by

$$\langle f,g\rangle = \int_0^1 f(x)g(x)\,dx.$$

Let $W \subseteq V$ be the subspace spanned by the following three functions: $f_0(x) = 1$, $f_1(x) = x$, and $f_2(x) = x^2$.

(a) Calculate the inner products $\langle f_i, f_j\rangle$ for all $i,j \in \{0,1,2\}$.

(b) Using the Gram-Schmidt method starting from $\{f_0, f_1, f_2\}$, find an orthonormal basis for W.

(c) **Approximation.** Consider the function g on $[0,1]$ defined by $g(x) = x^3$. Find the best quadratic approximation of g, i.e., find the quadratic function $h \in W$ such that

$$\int_0^1 (h(x) - g(x))^2\,dx$$

is as small as possible. Hint: this is equivalent to requiring that $\|h - g\|$ is as small as possible, i.e., h is the orthogonal projection of g onto the subspace W.

The following problems are additional proof drills.

Problem 5. Let $f : V \to W$ be a linear function, and assume f is one-to-one. Let $v_1,\ldots,v_n \in V$ be linearly independent. Prove that $f(v_1),\ldots,f(v_n)$ are linearly independent.

Problem 6. Let $f : V \to W$ be a linear function, and assume $v_1,\ldots,v_n \in V$ are points such that $f(v_1),\ldots,f(v_n)$ are linearly independent. Prove that $v_1,\ldots,v_n$ are linearly independent.

Problem 7. Let $f : V \to W$ be a linear function. Prove that $\ker f$ is a subspace of V. Also prove that $\mathrm{Im}\, f$ is a subspace of W.

Problem 8. Let $f : V \to W$ be a linear function, and let $U \subseteq V$ be a subspace of V. Recall the definition of direct image:

$$f(U) = \{w \in W \mid \text{there exists } u \in U \text{ with } f(u) = w\}.$$

Prove that $f(U)$ is a subspace of W.

Problem 9. Let $f : V \to W$ be a linear function, let $v_1,\ldots,v_m \in V$ be a basis of the kernel of f, and let $w_1,\ldots,w_p \in W$ be a basis of the image of f. Let $u_1,\ldots,u_p \in V$ be vectors such that $f(u_1) = w_1,\ldots,f(u_p) = w_p$. Prove that $\{v_1,\ldots,v_m,u_1,\ldots,u_p\}$ is a basis of V.