Math 4680/5680, Topics in Logic and Computation, Winter 2017

Handout 4: Problems

Problem 1. Show that the theory of dense linear order without endpoints is not categorical in the cardinality of the continuum.

Recall the definition of *homomorphism* and *isomorphism* of structures (Definition 4.3.1). The following theorem is obvious and can be proved by induction on terms and formulas:

Theorem (Isomorphism theorem)**.** *Let* $h : \mathfrak{A} \to \mathfrak{B}$ *be an isomorphism of structures. Then for any first-order formula* $\varphi(x_1, \ldots, x_n)$ *and any elements* $a_1, \ldots, a_n \in |\mathfrak{A}|$*, we have*

$$\mathfrak{A} \models \varphi(\bar{a}_1, \ldots, \bar{a}_n) \iff \mathfrak{B} \models \varphi(\overline{h(a_1)}, \ldots, \overline{h(a_n)})$$

Problem 2. Let h be an *automorphism* of the structure $\mathfrak{A}$, i.e, an isomorphism from $\mathfrak{A}$ to itself. Let R be an n-ary relation on $|\mathfrak{A}|$ definable in $\mathfrak{A}$. Prove: for all $a_1, \ldots, a_n \in |\mathfrak{A}|$,

$$(a_1, \ldots, a_n) \in R \iff (h(a_1), \ldots, h(a_n)) \in R.$$

Hint: use the isomorphism theorem.

Problem 3. Prove that $\mathbb{R}$ and $\emptyset$ are the only subsets of the real line $\mathbb{R}$ that are definable in $(\mathbb{R}; <)$. Hint: use Problem 2.

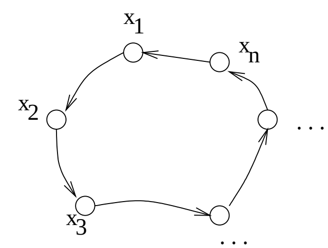
Problem 4. Let Γ, Δ be two sets of sentences in predicate logic. We say that Γ and Δ are *equivalent*, written $\Gamma \equiv \Delta$, if $\Gamma \vdash \varphi \iff \Delta \vdash \varphi$ for all sentences φ.

Prove: If $\Gamma \equiv \Delta$ are equivalent, and Δ is finite, then there exists some finite subset $\Gamma' \subseteq \Gamma$ such that $\Gamma' \equiv \Gamma$. Hint: let $\Delta = \{\varphi_1, \ldots, \varphi_n\}$, and consider $\bar{\Gamma} = \Gamma \cup \{\neg(\varphi_1 \wedge \ldots \wedge \varphi_n)\}$. What can you say about the consistency of $\bar{\Gamma}$?

Note: In this problem, indicate exactly the places (if any) where you use the soundness, completeness, and/or compactness theorems.

Problem 5. Prove: If the sentence σ holds in all infinite structures, then there exists some finite number n such that σ holds in all structures of n or more elements. Hint: let τ_n be a first-order sentence indicating that the structure has at least n distinct elements. Consider the consistency of the set $\{\neg\sigma, \tau_1, \tau_2, \tau_3, \ldots\}$.

Problem 6. Recall that the language of directed graphs has one binary predicate symbol $R(x, y)$, which we interpret as "there is an arrow from x to y". We say that a graph has a ***cycle*** if there exist elements $x_1, \ldots, x_n$, for some $n \geqslant 1$, such that there are arrows from x_1 to x_2 to x_3 to ... to x_n to x_1, as shown in the picture:



A graph is called ***cycle-free*** if it has no cycles. Prove that the class of cycle-free graphs is first-order axiomatizable, but not finitely axiomatizable. Hint: use the result from Problem 4.