Non-commutative k-spaces

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  - $(ab)^* = b^*a^*$ for all $a, b \in A$;
  - $(a + \lambda b)^* = a^* + \bar{\lambda}b^*$ for all $a, b \in A$ and $\lambda \in \mathbb{C}$.

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  - $*\colon A \to A$, $-\colon A \to A$, $+\colon A \times A \to A$, $\cdot\colon A \times A \to A$, $\cdot\colon \mathbb{C} \times A \to A$ are continuous.
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- A is a pro-C^*-algebra if it is Hausdorff, complete, and its topology coincides with $\mathcal{T}_A$.

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- Related notion: Locally convex approach $*$-algebra.

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- The tangent algebra of a C^*-algebra introduced by Arveson:
  - Universal factorizer of derivations into C^*-algebras;
  - Fails to be C^*-algebra, but it is a metrizable pro-C^*-algebra.

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  - $C(\omega_1) \to C(\omega_1 + 1)$ given by setting $f(\omega_1) = \lim_{x \to \omega_1} f(x)$ fails to be continuous.

Commutative case

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- $\Delta(A) =$ multiplicative functionals $\rho\colon A \longrightarrow \mathbb{C}$.
- $\Delta(A)$ is equipped with the w^*-topology – compact.
- Gelfand duality states:

$$A \longmapsto \Delta(A)$$
$$C(X) \longleftarrow X$$

is an equivalence of categories between commutative unital C^*-algebras and $\mathsf{CompHaus}^{\mathrm{op}}$.

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  - $\Delta(A_p) \to \Delta(A)$ is an embedding;
  - $\Delta(A) = \bigcup_{p \in \mathcal{N}(A)} \Delta(A_p)$;
  - for $f\colon \Delta(A) \to \mathbb{R}$, if $f_{|\Delta(A_p)}$ is continuous for every $p \in \mathcal{N}(A)$, then so is f.

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- $C_{\mathcal{F}}(X) = \mathcal{F}$-continuous maps $X \to \mathbb{C}$, with C^*-seminorms $p_F(f) = \sup_{x \in F} |f(x)|$, $F \in \mathcal{F}$.

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- $C_{\mathcal{F}}(X)$ is a pro-C^*-algebra.
- X is *strongly functionally generated* by $\mathcal{F}$ if every map in $C_{\mathcal{F}}(X)$ is continuous.

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(GL and El Harti, 2005/6) *The pair of functors*

$$A \longmapsto (\Delta(A), \Phi(A))$$
$$C_{\mathcal{F}}(X) \longleftarrow (X, \mathcal{F})$$

form an equivalence of categories between commutative unital pro-C^**-algebras and the opposite of a suitable category of s. f. g. Tychonoff spaces.*

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- (Dubuc and Porta, 1971) K-algebras $=$ $*$-algebra objects in kHaus ($+$ and $\cdot$ are k-continuous).
- Dubuc and Porta used k-ified compact-open topology for $\Delta(A)$. We use the w^*-topology (pointwise).

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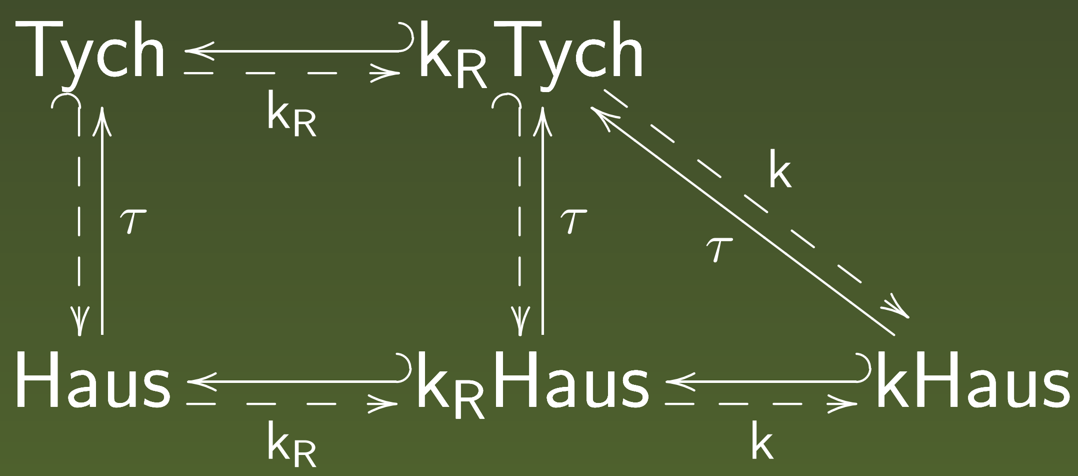

(The dashed arrows are right adjoints.)

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Pro-C^*-algebras are non-commutative $\mathsf{k_R}$Tych spaces.

Generalized Stone-Čech-compactification

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- $(-)_b \colon \overline{\mathsf{P}}_{\mathsf{d}} \longrightarrow \mathsf{C}$ *is a coreflector.*
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  - $\mathsf{C} = C^*$-algebras and $*$-homomorphisms.

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- $(-)_b \colon \overline{\mathsf{P}}_\mathsf{d} \longrightarrow \mathsf{C}$ *is a coreflector.*
- $(-)_b \colon \overline{\mathsf{P}}_\mathsf{d} \longrightarrow \mathsf{C}$ *is exact.*
- If I is a $*$-ideal and A/I is complete, then $(A/I)_b \cong A_b/I_b$.

Nicest properties of A_b

- $(A_b, \|\cdot\|_\infty)$ is a C^*-algebra.
- A_b is dense in A (in the topology of A).
- A_b depends only on the algebraic structure of A.

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- If A_b is simple, then A is a C^*-algebra.

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Further details / results: Bounded and Unitary Elements in Pro-C^*-algebras, *Appl. Categ. Structures*, **14** (2006), no. 2, 151–164.