Quasi-categories vs Segal spaces

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Two Model Structures on $\mathbf{S}$

If $i : A \to B$ and $f : X \to Y$, $i \pitchfork f$ means every commutative

$$\begin{array}{ccc} A & \xrightarrow{a} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow[b]{} & Y \end{array}$$

has a *diagonal filler* $d : B \to X$.

1)(Quillen) $h_n^k : \Lambda^k[n] \to \Delta[n]$, $f : X \to Y$ is a *Kan fibration* if $h_n^k \pitchfork f$, $0 \leq k \leq n$, $n \geq 1$. $\mathcal{F}_0$ = Kan fibrations.

$\mathbf{S}^{\pi_0}(X, Y) = \pi_0(Y^X)$. $u : A \to B$ is a *weak homotopy equivalence* if

$$\mathbf{S}^{\pi_0}(u, X) : \mathbf{S}^{\pi_0}(B, X) \to \mathbf{S}^{\pi_0}(A, X)$$

is a bijection for each Kan complex X. $\mathcal{W}_0$ = weak homotopy equivalences. $\mathcal{C}_0$ = monomorphisms.

$(\mathcal{F}_0, \mathcal{C}_0, \mathcal{W}_0)$ is the *classical model structure* on $\mathbf{S}$.

2)(Joyal) $\tau_1(X)$ is the *fundamental category* of X. $\tau_0(X) =$ isomorphism classes of objects in $\tau_1(X)$. $X \in \mathbf{S}$ is a *quasi-category* if $h^k_n \pitchfork X$, $0 < k < n$.

$\mathbf{S}^{\tau_0}(X,Y) = \tau_0(Y^X)$. $u : A \to B$ is a *weak categorical equivalence* if

$$\mathbf{S}^{\tau_0}(u,X) : \mathbf{S}^{\tau_0}(B,X) \to \mathbf{S}^{\tau_0}(A,X)$$

is a bijection for each quasi-category X. $\mathcal{W}_1 =$ weak categorical equivalences. $\mathcal{C}_1 =$ monomorphisms. $\mathcal{F}_1 = (\mathcal{C}_1 \bigcap \mathcal{W}_1)^{\pitchfork}$ the *quasi-fibrations*.

$(\mathcal{F}_1, \mathcal{C}_1, \mathcal{W}_1)$ is the *quasi-category model structure* on $\mathbf{S}$. The fibrant objects are the quasi-categories.

Note: $\mathcal{C}_0 = \mathcal{C}_1$ and $\mathcal{W}_1 \subseteq \mathcal{W}_0$ so $\mathcal{C}_1 \bigcap \mathcal{W}_1 \subseteq \mathcal{C}_0 \bigcap \mathcal{W}_0$ and $\mathcal{F}_0 \subseteq \mathcal{F}_1$

Bisimplicial Sets

$A, B \in \mathbf{S}$, $A \square B \in \mathbf{S^{(2)}}$ $(A\square B)_{mn} = A_m \times B_n$.
The *left divsion functor*: $\Delta[n] \to A\backslash X = A\square\Delta[n] \to X$.
The *right divsion functor*: $\Delta[m] \to X/B = \Delta[m]\square B \to X$.
Example: $\Delta[m]\backslash X = X_{m*}, X/\Delta[n] = X_{*n}$.

Extension to arrows: $u : A \to B, v : C \to D$ in $\mathbf{S}$

$$\begin{array}{ccc} A\square C & \longrightarrow & B\square C \\ \downarrow & & \downarrow \\ A\square D & \longrightarrow & B\square D \end{array}$$

gives $u\square' v : A\square D +_{A\square C} B\square C \to B\square D$.
For $f : X \to Y$ in $\mathbf{S^{(2)}}$

$$\begin{array}{ccc} B\backslash X & \longrightarrow & A\backslash X \\ \downarrow & & \downarrow \\ B\backslash Y & \longrightarrow & A\backslash Y \end{array}$$

gives $< u\backslash f >: B\backslash X \to B\backslash Y \times_{A\backslash Y} A\backslash X$.

$$\begin{CD} X/D @>>> X/C \\ @VVV @VVV \\ Y/D @>>> Y/C \end{CD}$$

gives $< f/v >: X/D \to Y/D \times_{Y/C} X/C$.

$(u\Box' v) \pitchfork f$ iff $v \pitchfork < u\backslash f >$ iff $u \pitchfork < f/v >$.

$i_2 : \Delta \to \Delta \times \Delta$ is $i_2([n])) = ([0],[n])$
$i_2^*(X) = X_0 = \Delta[0]\backslash X$, and

$$Hom_2(X,Y) = i_2^*(Y^X)$$

is an enrichment of $\mathbf{S^{(2)}}$ over $\mathbf{S}$. There are tensor and cotensor products.

The vertical model structure on $\mathbf{S}^{(2)}$

$f : X \to Y$ in $\mathbf{S}^{(2)}$ is a *vertical weak homotopy equivalence* if $f_m : X_m \to Y_m$ is a weak homotopy equivalence $m \geq 0$. $\mathcal{W}'_0 =$ class of all such. $s_m : \partial\Delta[m] \to \Delta[m]$ is the inclusion. $f : X \to Y$ is a *vertical fibration* or *v-fibration* if $< s_m \backslash f >$ is a Kan fibration $m \geq 0$. X is *v-fibrant* if $X \to 1$ is a v-fibration. $\mathcal{F}'_0 =$ class of v-fibrations. $\mathcal{C}'_0 =$ monomorphisms.

Theorem

$(\mathcal{F}'_0, \mathcal{C}'_0, \mathcal{W}'_0)$ is a simplicial model structure on $\mathbf{S}^{(2)}$ which is proper and cartesian closed.

This is the *Reedy model structure* associated to the classical model structure $(\mathcal{F}_0, \mathcal{C}_0, \mathcal{W}_0)$ on $\mathbf{S}$. We call it the *vertical model structure* on $\mathbf{S}^{(2)}$. There is also a horizontal model structure on $\mathbf{S}^{(2)}$ associated to the quasi-category model structure $(\mathcal{F}_1, \mathcal{C}_1, \mathcal{W}_1)$ on $\mathbf{S}$.

Note: $f : X \to Y$ in $\mathbf{S}^{(2)}$ is a vertical weak homotopy equivalence iff $Hom_2(f, Z) : Hom_2(Y, Z) \to Hom_2(X, Z)$ is a weak homotopy equivalence for each v-fibrant Z in $\mathbf{S}^{(2)}$.

Complete Segal Spaces

The $n-chain$ $I_n = \bigcup_{i=0}^{n-1}(i, i+1)$. $i_n : I_n \to \Delta[n]$ is the inclusion. $I_0 = 0$. $X \in \mathbf{S}^{(2)}$ satisfies the *Segal condition* if

$$i_n \backslash X : \Delta[n] \backslash X \to I_n \backslash X$$

is a weak homotopy equivalence for $n \geq 2$.
$I_n \backslash X = X_1 \times_{X_0} X_1 \times \ldots \times_{X_0} X_1$. So X satisfies the Segal condition iff the map

$$X_n \to X_1 \times_{X_0} X_1 \times \ldots \times_{X_0} X_1$$

is a weak homotopy equivalence for $n \geq 2$. Example: the nerve of a simplicial category - exactly. The name is from Graham Segal's Δ-spaces - the above with $X_0 = pt$. A *Segal space* is a v-fibrant simplicial space that satisfies the Segal condition. Introduced by Charles Rezk in his paper "A model for the homotopy theory of homotopy theory".

J is the nerve of the groupoid with one isomorphism $0 \to 1$. A Segal space X is *complete* if the map

$$1 \backslash X \to J \backslash X$$

is a weak homotopy equivalence. $f : X \to Y$ in $\mathbf{S}^{(2)}$ is a *Rezk weak equivalence* if

$$Hom_2(f, Z) : Hom_2(Y, Z) \to Hom_2(X, Z)$$

is a weak homotopy equivalence for each complete Segal space Z. $\mathcal{W}_R =$ Rezk weak equivalences. $\mathcal{C}_R =$ monomorphisms. $\mathcal{F}_R = (\mathcal{C}_R \bigcap \mathcal{W}_R)^{\pitchfork}$. Then Rezk proved

Theorem

$(\mathcal{F}_R, \mathcal{C}_R, \mathcal{W}_R)$ is a simplicial model structure on $\mathbf{S}^{(2)}$ which is left proper and cartesian closed. The fibrant objects are the complete Segal spaces.

$(\mathcal{F}_R, \mathcal{C}_R, \mathcal{W}_R)$ is the *Rezk model structure* or the *model structure for complete Segal spaces.*

Note: $\mathcal{C}'_0 = \mathcal{C}_R$ and $\mathcal{W}'_0 \subseteq \mathcal{W}_R$, so $\mathcal{F}_R \subseteq \mathcal{F}'_0$.

$i_1 : \Delta \to \Delta \times \Delta$ is $i_1([n])) = ([n], [0])$. $p_1 : \Delta \times \Delta \to \Delta$ is the first projection. $p_1 \dashv i_1$, so $p_1^* : \mathbf{S} \longleftrightarrow \mathbf{S^{(2)}} : i_1^*$. $i_1^*(X) = X_{*0}$ the first row of X, so $p_1^*(A) = A \Box 1$. Our main theorem is then

Theorem

$p_1^* : \mathbf{S} \longleftrightarrow \mathbf{S^{(2)}} : i_1^*$ is a Quillen equivalence between the model category for quasi-categories and the model category for complete Segal spaces.

Thus, all the homotopy theoretic information in a complete Segal space is contained in its first row.

$\Delta'[n]$ is the nerve of the groupoid freely generated by $[n]$. $t : \Delta \times \Delta \to \mathbf{S}$ is $t([m], [n]) = \Delta[m] \times \Delta'[n]$. $t_! : \mathbf{S^{(2)}} \to \mathbf{S}$ is the left Kan extension of t along $Yoneda : \Delta \times \Delta \to \mathbf{S^{(2)}}$. $t_!$ is the *total space* functor. It has a right adjoint $t^! : \mathbf{S} \to \mathbf{S^{(2)}}$

$$t^!(X)_{mn} = \mathbf{S}(\Delta[m] \times \Delta'[n], X)$$

Theorem

$t_! : \mathbf{S}^{(\mathbf{2})} \longleftrightarrow \mathbf{S} : t^!$ is a Quillen equivalence between the model category for complete Segal spaces and the model category for quasi-categories.

Note: $t_! p_1^* : \mathbf{S} \to \mathbf{S} = id_{\mathbf{S}}$ so $i_1^* t^! = id_{\mathbf{S}}$

Segal Categories

$X : \Delta^{op} \to \mathbf{S}$ is a *precategory* if X_0 is discrete. $\mathbf{PCat} \subseteq \mathbf{S}^{(\mathbf{2})}$ is the full subcategory of precategories. $X : (\Delta \times \Delta)^{op} \to \mathbf{Set}$ is in $\mathbf{PCat}$ iff it takes every map in $[0] \times \Delta$ to a bijection, so put $\Delta^{|2} = ([0] \times \Delta)^{-1}(\Delta \times \Delta)$ and let $\pi : \Delta^2 \to \Delta^{|2}$ be the canonical map.

$$\pi^* : [(\Delta^{|2})^{op}, \mathbf{Set}] \simeq \mathbf{PCat} \subseteq \mathbf{S}^{(\mathbf{2})}$$

$X \in \mathbf{PCat}$ is a *Segal category* if it satisfies the Segal condition. Segal categories were introduced by Hirshowitz and Simpson for applications to algebraic geometry. They showed

Theorem

There is a model structure on **PCat** *in which the cofibrations are the monomorphisms and the weak equivalences are "weak categorical equivalences". The model structure is left proper and cartesian closed.*

This is the *Hirshowitz-Simpson model structure*, or the *model structure for Segal categories*.

Julia Bergner showed

Theorem

The adjoint pair $\pi^* : \mathbf{PCat} \longleftrightarrow \mathbf{S^{(2)}} : \pi_*$ *is a Quillen equivalence between the model category for Segal categories and the model category for complete Segal spaces. A map* $f : X \to Y$ *of precategories is a weak categorical equivalence iff* $\pi^*(f)$ *is a Rezk weak equivalence.*

$p_1 : \Delta \times \Delta \longleftrightarrow \Delta : i_1$ and p_1 inverts the arrows of $[0] \times \Delta$, so there is a unique $q : \Delta^{|2} \to \Delta$ such that $q\pi = p_1$.
$j = \pi i_1 : \Delta \to \Delta^{|2}$ satisfies $q \dashv j$. If $X \in \mathbf{PCat}$, $j^*(X) = X_{*0}$ - the first row of X. If $A \in \mathbf{S}$, $q^*(A) = A \square 1$.

Theorem

The adjoint pair $q^* : \mathbf{S} \longleftrightarrow \mathbf{PCat} : j^*$ is a Quillen equivalence between the model category for quasi-categories and the model category for for Segal categories.

Put $d = \pi\delta : \Delta \to \Delta^{|2}$, where $\delta : \Delta \to \Delta \times \Delta$ is the diagonal. If $X \in \mathbf{PCat}$, $d^*(X) =$ the diagonal complex of X. d^* has a left adjoint $d_!$ and a right adjoint d_*.

Theorem

The adjoint pair $d^* : \mathbf{PCat} \longleftrightarrow \mathbf{S} : d_*$ is a Quillen equivalence between the model category for Segal categories and the model category for quasi-categories.

$d^* q^* : \mathbf{S} \to \mathbf{S} = id_{\mathbf{S}}$ since $qd = id$. Hence $j^* d_* : \mathbf{S} \to \mathbf{S} = id_{\mathbf{S}}$.