

Colimits and Profunctors

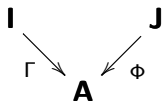
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The Problem

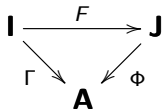
For two diagrams



what is the most general kind of morphism $\Gamma \rightsquigarrow \Phi$ which will produce a morphism

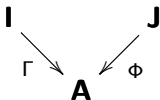
$$\lim_{\rightarrow} \Gamma \longrightarrow \lim_{\rightarrow} \Phi \quad ?$$

Trivial answer: A morphism $\lim_{\rightarrow} \Gamma \longrightarrow \lim_{\rightarrow} \Phi$.
We want something more syntactic! E.g.



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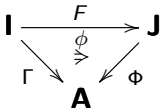
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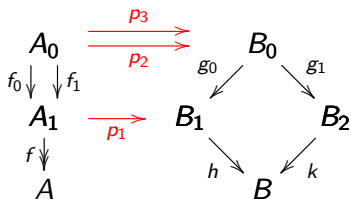
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Example



$$p_1 f_0 = g_0 p_2$$

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$$g_1 p_2 = g_1 p_3$$

Then we get

$$h p_1 f_0 = h g_0 p_2$$

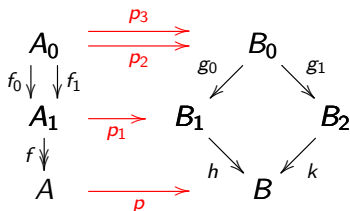
$$= h g_1 p_2$$

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$$= h g_0 p_3$$

$$= h p_1 f_1$$

Example



$$p_1 f_0 = g_0 p_2$$

$$p_1 f_1 = g_0 p_3$$

$$g_1 p_2 = g_1 p_3$$

Thus we get

$$\begin{aligned} hp_1 f_0 &= hg_0 p_2 \\ &= kg_1 p_2 \\ &= kg_1 p_3 \\ &= hg_0 p_3 \\ &= hp_1 f_1 \end{aligned}$$

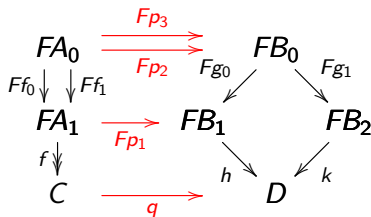
So there is a unique p such that $pf = hp_1$.

Problems

- ▶ Different schemes (number of arrows, placement, equations) may give the same p
- ▶ It might be difficult to compose such schemes

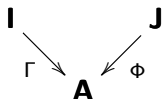
On the positive side

- ▶ It is equational so for any functor $F : \mathbf{A} \longrightarrow \mathbf{B}$ for which the coequalizer and pushout below exist we get an induced morphism q



The Problem (Refined)

For two diagrams in \mathbf{A}



what is the most general kind of morphism $\Gamma \rightsquigarrow \Phi$ which will produce a morphism

$$\varinjlim F\Gamma \longrightarrow \varinjlim F\Phi$$

for every $F : \mathbf{A} \longrightarrow \mathbf{B}$ for which the \varinjlim 's exist?

- ▶ It should be natural in F (in a way to be specified)

First Solution

Take F to be the Yoneda embedding $Y : \mathbf{A} \longrightarrow \mathbf{Set}^{\mathbf{A}^{op}}$. Then we have the bijections

$$\frac{\frac{\frac{\lim_{\rightarrow} Y\Gamma \longrightarrow \lim_{\rightarrow} Y\Phi}{\lim_{\rightarrow I} \mathbf{A}(-, \Gamma I) \longrightarrow \lim_{\rightarrow J} \mathbf{A}(-, \Phi J)}}{\langle \mathbf{A}(-, \Gamma I) \longrightarrow \lim_{\rightarrow J} \mathbf{A}(-, \Phi J) \rangle_I}}{\langle x_I \in \lim_{\rightarrow J} \mathbf{A}(\Gamma I, \Phi J) \rangle_I}$$

An element of $\lim_{\rightarrow J} \mathbf{A}(\Gamma I, \Phi J)$ is an equivalence class of morphisms

$$[\Gamma I \xrightarrow{a} \Phi J]_J$$

where $a \sim a'$ iff there is a path of diagrams

$$\begin{array}{ccc} \Gamma I & \xrightarrow{a_k} & \Phi J_k \\ \parallel & & \downarrow \Phi j_k \\ \Gamma I & \xrightarrow{a_{k+1}} & \Phi J_{k+1} \end{array}$$

joining a to a' .

Theorem

Suppose we are given

- ▶ For each I , a J_I and a morphism $\Gamma I \xrightarrow{a_I} \Phi J_I$
- ▶ For each $I' \xrightarrow{i} I$ a path of J 's and a 's joining

$$\Gamma I' \xrightarrow{\Gamma i} \Gamma I \xrightarrow{a_I} \Phi J_I$$

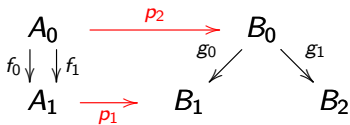
to

$$\Gamma I' \xrightarrow{a_{I'}} \Phi J_{I'}$$

then for every F we get a morphism $\varinjlim F\Gamma \rightarrow \varinjlim F\Phi$. Two such choices, $\langle a_I : \Gamma I \rightarrow \Phi J_I \rangle$ and $\langle a'_I : \Gamma I \rightarrow \Phi J'_I \rangle$, induce the same morphisms $\varinjlim F\Gamma \rightarrow \varinjlim F\Phi$, iff for each J there is a path joining

$$\Gamma I \xrightarrow{a_I} \Phi J_I \text{ to } \Gamma I \xrightarrow{a'_I} \Phi J'_I.$$

Example Again



$$\begin{array}{ccc} A_0 & \xrightarrow{f_0} & A_1 & \xrightarrow{p_1} & B_1 \\ \parallel & & & & \uparrow g_0 \\ A_0 & \xrightarrow{p_2} & & & B_0 \end{array}$$

$$\begin{array}{ccc} A_0 & \xrightarrow{f_1} & A_1 & \xrightarrow{p_1} & B_1 \\ \parallel & & & & \uparrow g_0 \\ A_0 & \xrightarrow{p_3} & & & B_0 \\ \parallel & & & & \downarrow g_1 \\ A_0 & \xrightarrow{\quad} & & & B_2 \\ \parallel & & & & \uparrow g_1 \\ A_0 & \xrightarrow{p_2} & & & B_0 \end{array}$$

Canonization

Recalling our first idea of

$$\begin{array}{ccc} \mathbf{I} & \xrightarrow{F} & \mathbf{J} \\ & \searrow \Gamma & \swarrow \Phi \\ & \mathbf{A} & \end{array}$$

ϕ

where we get for every I , a $J_I = FI$, and a morphism $a_I = \phi I : \Gamma I \rightarrow \Phi FI$. Naturality of ϕ gives a one-step path

$$\begin{array}{ccccc} \Gamma I' & \xrightarrow{\Gamma i} & \Gamma I & \xrightarrow{\phi I} & \Phi FI \\ \parallel & & & & \uparrow \phi Fi \\ \Gamma I' & \xrightarrow{\phi I'} & & & \Phi FI' \end{array}$$

In the general case $I \rightsquigarrow J_I$ is not a functor. There can be several J_I , and for $i : I' \rightarrow I$ we don't get a morphism $J_{I'} \rightarrow J_I$ but only a path. This is a kind of "relation between categories". They are called profunctors (distributors, bimodules, modules, relators).

Profunctors

- ▶ A *profunctor* $P : \mathbf{A} \multimap \mathbf{B}$ is a functor $P : \mathbf{A}^{op} \times \mathbf{B} \rightarrow \mathbf{Set}$
- ▶ Every functor $F : \mathbf{A} \rightarrow \mathbf{B}$ gives two profunctors

$$F_* : \mathbf{A} \multimap \mathbf{B}, \quad F_* = \mathbf{B}(F-, -) : \mathbf{A}^{op} \times \mathbf{B} \rightarrow \mathbf{Set}$$

$$F^* : \mathbf{B} \multimap \mathbf{A}, \quad F^* = \mathbf{B}(-, F-) : \mathbf{B}^{op} \times \mathbf{A} \rightarrow \mathbf{Set}$$

$$F_* \dashv F^*$$

- ▶ Composition $\mathbf{A} \xrightarrow{P} \mathbf{B} \xrightarrow{Q} \mathbf{C}$

$$Q \otimes P(A, C) = \int^B Q(B, C) \times P(A, B)$$

$$= \{[A \xrightarrow[x]{P} B \xrightarrow[y]{Q} C]_B\} = \{y \otimes_B x\}$$

- ▶ $A \xrightarrow{x} B \xrightarrow{y} C \sim A \xrightarrow{x'} B' \xrightarrow{y'} C$ if there is

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \\ \parallel & & \downarrow b & & \parallel \\ A & \xrightarrow{x'} & B' & \xrightarrow{y'} & C \end{array}$$

$$\begin{aligned} y \otimes x &= y' b \otimes x \\ &= y' \otimes bx \\ &= y' \otimes x' \end{aligned}$$

For example, given functors

$$\mathbf{I} \xrightarrow{\Gamma} \mathbf{A} \xleftarrow{\Phi} \mathbf{J}$$

we get an easily computed profunctor $\Phi^* \otimes \Gamma_* : \mathbf{I} \dashrightarrow \mathbf{J}$

$$\Phi^* \otimes \Gamma_*(I, J) = \mathbf{A}(\Gamma I, \Phi J).$$

Proposition

A compatible family $\langle x_I \in \varinjlim_J \mathbf{A}(\Gamma I, \Phi J) \rangle_J$ determines a subprofunctor $P \subseteq \Phi^* \otimes \Gamma_*$ with the property that for every F and every $a \in P(I, J)$ we have

$$\begin{array}{ccc} F\Gamma I & \xrightarrow{Fa} & F\Phi J \\ \text{inj}_I \downarrow & & \downarrow \text{inj}_J \\ \varinjlim F\Gamma & \longrightarrow & \varinjlim F\Phi \end{array}$$

for the morphism induced by $\langle x_I \rangle$.

Proof.

$$P(I, J) = \{a : \Gamma I \longrightarrow \Phi J \mid [a] = [x_I]\}.$$



Total Profunctors

Definition

$P : \mathbf{A} \multimap \mathbf{B}$ is *total* if for every A ,

$$\lim_{\rightarrow B} P(A, B) \cong 1.$$

Let $T : \mathbf{A} \rightarrow \mathbf{1}$ be the unique functor. Then P is total iff

$$T_* \otimes P \xrightarrow{\cong} T_*.$$

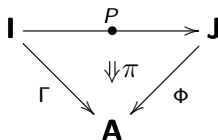
Proposition

- (1) Total profunctors are closed under composition.
- (2) For any functor $F : \mathbf{A} \rightarrow \mathbf{B}$, F_* is total. (In particular Id_A is total.)
- (3) If P and $P \otimes Q$ are total then Q is total.
- (4) Total profunctors are closed under connected colimits and quotients.
- (5) F^* is total iff F is final.
- (6) For $\mathbf{I} \xleftarrow{\Sigma} \mathbf{K} \xrightarrow{\Theta} \mathbf{J}$, $\Theta_* \otimes \Sigma^*$ is total iff Σ is final.

Profunctors over \mathbf{A}

Definition

For $\Gamma : \mathbf{I} \rightarrow \mathbf{A}$ and $\Phi : \mathbf{J} \rightarrow \mathbf{A}$, a profunctor from Γ to Φ (or a profunctor from \mathbf{I} to \mathbf{J} over \mathbf{A}) is



where P is a profunctor $\mathbf{I} \rightarrow \bullet \rightarrow \mathbf{J}$ and

$\pi : P \rightarrow \mathbf{A}(\Gamma -, \Phi -) = \Phi^* \otimes \Gamma_*$ is a natural transformation.

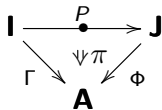
Profunctors over \mathbf{A} compose in the “obvious” way:

$$(Q, \psi) \otimes (P, \pi) = (Q \otimes P, \psi \otimes \pi)$$

$$\psi \otimes \pi(y \otimes x) = (\psi y)(\pi x).$$

Theorem

Let



be a profunctor over \mathbf{A} with P total. Then for every $F : \mathbf{A} \rightarrow \mathbf{B}$ for which $\lim_{\rightarrow} F\Gamma$ and $\lim_{\rightarrow} F\Phi$ exist, there is a unique morphism $\lim_{\rightarrow} F\pi : \lim_{\rightarrow} F\Gamma \rightarrow \lim_{\rightarrow} F\Phi$ such that for every $x \in P(I, J)$ we have

$$\begin{array}{ccc}
 F\Gamma I & \xrightarrow{F\pi(x)} & F\Phi J \\
 \text{inj}_I \downarrow & & \downarrow \text{inj}_J \\
 \lim_{\rightarrow} F\Gamma & \xrightarrow{\lim_{\rightarrow} F\phi} & \lim_{\rightarrow} F\Phi
 \end{array}$$

If $(Q, \psi) : \Phi \rightarrow \Psi$ is another total profunctor over \mathbf{A} , we have

$$\lim_{\rightarrow} F(\psi \otimes \pi) = (\lim_{\rightarrow} F\psi)(\lim_{\rightarrow} F\pi).$$

Saturation

Definition

$P \twoheadrightarrow Q : \mathbf{I} \longrightarrow \mathbf{J}$ is *saturated* if $x \in Q(I, J)$ and for some $j : J \longrightarrow J'$, $jx \in P(I, J')$ implies $x \in P(I, J)$.

- ▶ P is saturated in Q iff for every I , $P(I, -) \twoheadrightarrow Q(I, -)$ is complemented in $\mathbf{Set}^{\mathbf{J}}$.
- ▶ Every $P \twoheadrightarrow Q$ has a saturation $\bar{P} \twoheadrightarrow Q$.

Theorem

Let (P, π) and (P', π') be two total profunctors $\Gamma \longrightarrow \Phi$. Then they induce the same family $\varinjlim F\Gamma \longrightarrow \varinjlim F\Phi$ iff the images of $\pi : P \longrightarrow \Phi^* \otimes \Gamma_*$ and $\pi' : P' \longrightarrow \Phi^* \otimes \Gamma_*$ have the same saturation.

Naturality

Definition

A family of morphisms $b_F : \varinjlim F\Gamma \longrightarrow \varinjlim F\Phi$ is natural if for every G we have

$$\begin{array}{ccc} \varinjlim GF\Gamma & \xrightarrow{b_{GF}} & \varinjlim GF\Phi \\ \downarrow & & \downarrow \\ G \varinjlim F\Gamma & \xrightarrow{Gb_F} & G \varinjlim F\Phi \end{array}$$

Theorem

A total profunctor over \mathbf{A} induces a natural family as above. Every natural family comes from a total saturated profunctor $\subseteq \Phi^ \otimes \Gamma_*$. In fact there is a bijection between natural families and saturated total $\subseteq \Phi^* \otimes \Gamma_*$.*

Cohesive Families

As remarked by Bénabou already in the 70's, a category over \mathbf{I}

$$\begin{array}{c} \mathbf{K} \\ \Lambda \downarrow \\ \mathbf{I} \end{array}$$

corresponds to a lax normal functor $\mathbf{I} \longrightarrow \mathbf{Prof}$ where an object I is sent to \mathbf{K}_I , the fibre over I , and a morphism $i : I \longrightarrow I'$ to the profunctor $P_i : \mathbf{K}_I \dashrightarrow \mathbf{K}_{I'}$ given by the formula

$$P_i(K, K') = \{K \xrightarrow{k} K' \mid \Lambda k = i\}$$

Definition

$\Lambda : \mathbf{K} \longrightarrow \mathbf{I}$ is a *cohesive* family of categories if each P_i is total.

Cohesive Families (Continued)

In elementary terms, for every K in \mathbf{K} and every morphism $i : \Lambda K \rightarrow I'$, there exists a morphism $k : K \xrightarrow{k} K'$ such that $i = \Lambda k$ and any two such liftings are connected by a path over i .

$$K \xrightarrow{\quad k \quad} K'$$

$$\Lambda K \xrightarrow{\quad i \quad} I'$$

Proposition

- (1) *Opfibrations are cohesive families*
- (2) *Cohesive families are stable under pullback*
- (3) *Cohesive families are closed under composition*

Cohesive Families of Diagrams

Definition

A *cohesive family of diagrams* in \mathbf{A} is a span

$$\begin{array}{ccc} \mathbf{K} & \xrightarrow{\Gamma} & \mathbf{A} \\ \Lambda \downarrow & & \\ \mathbf{I} & & \end{array}$$

with Λ cohesive.

Let $\Gamma_I = \Gamma|_{\mathbf{K}_I}$.

Theorem

$\lim_{\rightarrow} \Gamma_I$ extends to a unique functor $\lim_{\rightarrow} \Gamma(_) : \mathbf{I} \rightarrow \mathbf{A}$ such that for all $k : K \rightarrow K'$ over $i : I \rightarrow I'$

$$\begin{array}{ccc} \Gamma K & \xrightarrow{\Gamma k} & \Gamma K' \\ \text{inj}_K \downarrow & & \downarrow \text{inj}_{K'} \\ \lim_{\rightarrow} \Gamma_I & \xrightarrow{\lim_{\rightarrow} \Gamma_i} & \lim_{\rightarrow} \Gamma_{I'} \end{array}$$

Kan Extensions

$\lim_{\rightarrow} \Gamma(_) : \mathbf{I} \rightarrow \mathbf{A}$ is the left Kan extension and cohesiveness says it is fibrewise. So a more functorial version of the preceding theorem is:

Theorem

$\Lambda : \mathbf{K} \rightarrow \mathbf{I}$ is cohesive iff for every pullback diagram

$$\begin{array}{ccc} \mathbf{L} & \xrightarrow{G} & \mathbf{K} \\ \Sigma \downarrow & & \downarrow \Lambda \\ \mathbf{J} & \xrightarrow{F} & \mathbf{I} \end{array}$$

and every cocomplete \mathbf{A} , the canonical morphism

$$\begin{array}{ccc} \mathbf{A}^{\mathbf{L}} & \xleftarrow{G^*} & \mathbf{A}^{\mathbf{K}} \\ \text{Lan}_{\Sigma} \downarrow & \Downarrow \lambda & \downarrow \text{Lan}_{\Lambda} \\ \mathbf{A}^{\mathbf{J}} & \xleftarrow{F^*} & \mathbf{A}^{\mathbf{I}} \end{array}$$

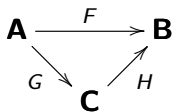
is an isomorphism.

If we take $\mathbb{J} = \mathbf{1}$, $F \leftrightarrow I \in \mathbf{I}$, we get $(\text{Lan}_{\Lambda} \Gamma)I \cong \lim_{\rightarrow} \Gamma_I$.

The Comprehensive Factorization

Set		Cat
Relations	\leftrightarrow	Profunctors
Everywhere Defined	\leftrightarrow	Total
Single Valued	\leftrightarrow	?
Functions	\leftrightarrow	Functors

Recall the *comprehensive factorization* on **Cat** (Street & Walters '73). Every functor F factors as

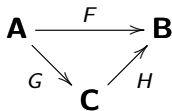


with G final and H a discrete fibration. So the final functors are “epi-like” and the discrete fibrations are “mono-like”.

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Discrete Valued Profunctors

Definition

P is *discrete valued* if it is of the form $P \cong G_* \otimes F^*$ for some $\mathbf{A} \xleftarrow{F} \mathbf{C} \xrightarrow{G} \mathbf{B}$ with F a discrete fibration.

Theorem

P is *discrete valued* iff for every A , $P(A, -)$ is *multirepresentable* (Diers), i.e. a sum of representables. In fact

$$P(A, -) \cong \sum_{FC=A} \mathbf{B}(GC, -).$$

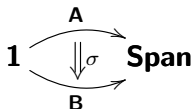
Corollary

The factorization $P \cong G_* \otimes F^*$ is unique up to isomorphism.

Mealy Morphisms

A small category is a monad in **Span**, which is a lax functor $\mathbf{1} \rightarrow \mathbf{Span}$.

A lax transformation



corresponds to a Mealy morphism (machine)

- ▶ For every A, B we are given a set $S(A, B)$ of *states*
- ▶ Arrows of \mathbf{A} are the input alphabet
- ▶ Arrows of \mathbf{B} are the output alphabet
- ▶ Action

$$A' \xrightarrow{a} A \xrightarrow{\bullet \xrightarrow{s}} B \quad \xrightarrow{\sigma} \quad A' \xrightarrow{\bullet \xrightarrow{s^a}} B' \xrightarrow{\sigma(s,a)} B$$

Mealy Profunctors

A Mealy morphism determines a profunctor $P : \mathbf{A} \dashrightarrow \mathbf{B}$

$$P(A, B) = \sum_{s \in S(A, B')} \mathbf{B}(B', B)$$

Theorem

P is a Mealy profunctor iff P is discrete valued.

Theorem

P is representable iff it is total and discrete valued.