On Tannaka Dualities

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What Tannaka duality is.

What I do.

What should be done.
Introduction

What Tannaka duality is.

- Tannaka duality is a duality between algebraic structures and their representations.
- Tannaka duality consists of reconstruction and representation.

What I do.

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- Tannaka duality consists of reconstruction and representation.

What I do.

- Reconstruct a Hopf algebra in Rel from its representations.
- Estimate the number of monoidal structures on the category of automata.

What should be done.

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- Tannaka duality consists of reconstruction and representation.

What I do.

- Reconstruct a Hopf algebra in $\text{Rel}$ from its representations.
- Estimate the number of monoidal structures on the category of automata.

What should be done.

- On fundamental theorem.
- On representation problem.
1. Tannaka Duality Theorem
Some references on Tannaka duality theorem and its generalizations.

- A. Joyal and R. Street, *An introduction to Tannaka duality and Quantum groups*.
- P. McCrudden, *Tannaka duality for Maschkean categories*.
- P. Deligne and J.S. Milne, *Tannakian Categories*.
Tannaka duality in \( \text{Vect}_k \)

Taking representations

Given a coalgebra \( C \) in \( \text{Vect}_k \), one can construct the category \( \text{Rep}_f(C) \) of finite dimensional representations of \( C \). Denote the forgetful functor by \( F_C : \text{Rep}_f(C) \to \text{Vect}_k \).

**Remark:** representations of \( C \) = right \( C \)-comodules.

\[
\text{Constructively, this is constructed by taking an appropriate quotient space:} \quad C_F = \left( \bigoplus \text{Rep}_f(C) \otimes F(C) \right) = \sim \quad (1)
\]
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Converse construction

Given \( F : C \to \text{Vect}_k \), a functor s.t. \( F(A) \) is finite dimensional, one can construct \( C_F \in \text{Vect}_k \), the coalgebra obtained by:

\[
C_F = \int_{\tau \in C} F(\tau)^* \otimes F(\tau) \tag{1}
\]
Tannaka duality in $\text{Vect}_k$

Taking representations

Given a coalgebra $C$ in $\text{Vect}_k$, one can construct the category $\text{Rep}_f(C)$ of finite dimensional representations of $C$. Denote the forgetful functor by $F_C : \text{Rep}_f(C) \to \text{Vect}_k$.

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Converse construction

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$$C_F = \int_{\tau \in C} F(\tau)^* \otimes F(\tau) \quad (1)$$

Constructively, this is constructed by taking an appropriate quotient space:

$$C_F = \left( \bigoplus_{\tau \in C} F(\tau)^* \otimes F(\tau) \right) / \sim \quad (2)$$
Fundamental Theorem of Coalgebras

A coalgebra in $\text{Vect}_k$ is the union of its finite dimensional sub-coalgebras.

This is essentially because vectors in $C \otimes C$ is a finite sum of $c_1 \otimes c_2$. 
Tannaka duality in $\text{Vect}_k$

**Fundamental Theorem of Coalgebras**

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**Theorem (Reconstruction theorem)**

For an arbitrary coalgebra $C \in \text{Vect}_k$, if $F : C \rightarrow \text{Vect}_k$ is the forgetful functor $F_C : \text{Rep}_f(C) \rightarrow \text{Vect}_k$, then we have an isomorphism:

$$C \cong C_{F_C}$$ (3)

**Coend formula**

A coalgebra can be reconstructed from its finite dimensional representations:

$$C = \int_{\tau \in \text{Rep}_f(C)} F(\tau)^* \otimes F(\tau)$$
There is a canonical functor $\bar{F} : \mathbf{C} \to \text{Rep}_f(C_F)$ such that the following commutes:

$$
\begin{array}{ccc}
\mathbf{C} & \xrightarrow{\bar{F}} & \text{Rep}_f(C_F) \\
\downarrow{F} & & \downarrow{F_{CF}} \\
\text{Vect}_k & & \\
\end{array}
$$

Remarkably, there is a characterization of fibre functors $F : \mathbf{C} \to \text{Vect}_k$ such that $\bar{F} : \mathbf{C} \to \text{Rep}_f(C_F)$ is an equivalence.
Tannaka duality in $\text{Vect}_k$

**Comparison functor**

There is a canonical functor $\bar{F} : C \to \text{Rep}_f(C_F)$ such that the following commutes:

$$C \xrightarrow{\bar{F}} \text{Rep}_f(C_F) \xleftarrow{F_{CF}} \text{Vect}_k$$

Remarkably, there is a characterization of fibre functors $F : C \to \text{Vect}_k$ such that $\bar{F} : C \to \text{Rep}_f(C_F)$ is an equivalence.

**Theorem (Representation theorem)**

If $C$ is $k$-linear abelian and $F$ is exact and faithful, then $\bar{F}$ is an equivalence of categories (and vice versa).
Main theme of Tannaka duality can be decomposed into the following two parts:

- **Reconstruction problem:**
  to reconstruct an algebraic structure from the category of its representations.
  - compact groups [Tannaka, ’39], [Krein, ’49]
  - locally compact groups [Tatsuuma, ’67]
  - Hopf algebras [Ulbrich, ’91]
  - quasi Hopf algebras [Majid, ’92] etc.

- **Representation problem:**
  to characterize what category is equivalent to a category of representations of an algebraic structure.
  - pro-algebraic groups [Deligne and Milne, ’81] : Tannakian category
  - compact groups [Doplicher and Roberts, ’89]
The following universality of a coalgebra is important:

**Universality of Coalgebra**

\[ C = \int_{\tau \in \text{Rep}_f(C)} F(\tau)^* \otimes F(\tau) \]

because this universality shows several correspondences between structures on \( \text{Rep}_f(C) \) and those on \( C \).
Bialgebra structures induce monoidal structures.

**Multiplication to monoidal structure**

Given a bialgebra structure \((\mu, \eta)\) on a coalgebra \(C \in \text{Vect}_k\), one can construct a monoidal structure \((\otimes_\mu, I_\eta)\) on \(\text{Rep}_f(C)\), s.t. the forgetful functor \(F_C : \text{Rep}_f(C) \rightarrow \text{Vect}_k\) is monoidal.

Remark: We mean strong monoidal by "monoidal".

Remark: The non-strong case is also studied in, e.g., [Majid, '92].
Tannaka duality in \( \text{Vect}_k \)

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Conversely, we have the inverse construction due to the universality of coalgebras.

**Monoidal structure to multiplication**

Given a functor \(F : C \to \text{Vect}_k\) and a monoidal structure \((\otimes, I)\) on \(C\) s.t. \(F\) is monoidal, one can construct a bialgebra structure \((\mu_\otimes, \eta_I)\) on \(C_F\).

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**Remark**: The non-strong case is also studied in, e.g., [Majid, ’92].
Antipodes induce left dual objects.

**Antipode to duals**

If a bialgebra $B \in \text{Vect}_k$ has its antipode $S : B \to B$, then the monoidal category $\text{Rep}_f(B)$ has left dual objects.
Antipodes induce left dual objects.

**Antipode to duals**

If a bialgebra $B \in \text{Vect}_k$ has its antipode $S : B \to B$, then the monoidal category $\text{Rep}_f(B)$ has left dual objects.

The converse is also true.

**Dual to antipode**

Given a monoidal functor $F : C \to \text{Vect}_k$ s.t. $C$ has left dual objects, then the bialgebra $C_F$ is a Hopf algebra.

**Especially..**

The monoidal category $\text{Rep}_f(B)$ has left dual objects if and only if $B$ is a Hopf algebra.
Some generalizations of Tannaka duality theorem

There are known several directions to generalize Tannaka duality theorem and its analogues.

- **Tannakian categories** [P. Deligne and J. Milne, ’82]
- **Tannaka duality for Maschkean categories** [P. McCrudden, ’02]
- **Enriched Tannaka reconstruction** [B. Day, ’96]
2. Discrete Analogue of Tannaka Duality
This study is originally aimed at solving the following classification problem.

**Original Problem**

- How many monoidal structures can exist on the category $\text{Aut}(\Sigma)$ of automata and simulations?
- Are there infinitely many monoidal structures?
- Can we give a good classification of them?
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The motivation comes from the following recent approach to concurrency theory based on categorical framework of state-based systems.

**Motivation**

*"The microcosm principle and concurrency in coalgebra"* [Jacobs et al, '08]

- Understand several existing constructions on state-based systems as categorical operations on particular category of universal coalgebra.
Tannaka duality in \textbf{Rel}: Reconstruction problem

Given a Hopf algebra $H \in \text{Rel}$, we have following universality of $H$.

(Almost trivial) Universality of $H$

\begin{equation}
H = \int_{\tau \in \text{Rep}(H)} F(\tau)^* \otimes F(\tau)
\end{equation}

But this expression is not satisfactory because \textbf{Rel} is neither complete nor cocomplete. In fact:
Tannaka duality in \textbf{Rel}: Reconstruction problem

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\]  

But this expression is not satisfactory because \textbf{Rel} is neither complete nor cocomplete. In fact:

\textbf{Lack of (co-) equalizers}

\( X = \{\bullet, \bullet\} \in \textbf{Rel} \) and consider the following relation \( f : X \to X \):

Then there is no equalizer for \( f \) and the identity \( \text{id}_X : X \to X \).
Reconstruction problem

Tannaka duality theorem (Reconstruction of compact groups)

$G$: compact group, $\text{Rep}_f(G, \mathbb{C})$: category of fin. dim. rep. of $G$.

$F : \text{Rep}_f(G, \mathbb{C}) \to \text{Vect}_k$ = the forgetful functor. Let $T(G) \subseteq \text{End}(F)$ be a subset of natural transformations $F \Rightarrow F$ satisfying:

$$
U(\tau \otimes \rho) = U(\tau) \otimes U(\rho)
$$

$$
U(I) = id_I
$$

$$
\overline{U} = U
$$

Then $T(G)$ forms a topological group and is canonically isomorphic to $G$.

Similar construction is known also for pro-algebraic groups [Deligne-Milne].
### Tannaka duality theorem (Reconstruction of compact groups)

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\]

Then $T(G)$ forms a topological group and is canonically isomorphic to $G$.

Similar construction is known also for pro-algebraic groups [Deligne-Milne].

### Reconstruction via natural transformations

Can we reconstruct $H \in \text{Rel}$ by using some class of natural transformations $F_H \Rightarrow F_H$ on the forgetful functor $F_H : \text{Rep}(H) \to \text{Rel}$?
Reconstruction problem

\( \mathbf{C} \): arbitrary monoidal category with left dual objects.
\( F : \mathbf{C} \to \text{Rel} \): a (strict) monoidal functor.

**Poset structure on** \( \text{End}(F) \)

Given \( U, V : F \Rightarrow F \), we denote by \( U \leq V \) if for each \( \tau \in \mathbf{C} \),

\[
U(\tau) \subseteq V(\tau)
\]

**Remark** : \( \text{End}(F) \ni U : F \Rightarrow F \) consists of \( U(\tau) \subseteq F(\tau) \times F(\tau) \).
Reconstruction problem

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**Conjugate operator on** \( \text{End}(F) \)

Given \( U \in \text{End}(F) \), the conjugate \( \bar{U} : F \Rightarrow F \) is defined: for each \( \tau \in \mathbf{C} \),
the component on \( \tau \) is given by,

\[
\bar{U}(\tau) = (U(\tau^*))^*
\]

**Remark**: The internal \( * \) is dual in \( \mathbf{C} \), and the external \( * \) is dual in \( \mathbf{Rel} \).
Reconstruction problem

Especially, there is the minimal element $0 : F \Rightarrow F$ whose components are empty sets $0(\tau) = \emptyset \subseteq F(\tau) \times F(\tau)$.

**Atoms in End$(F)$**

A natural transformation $U : F \Rightarrow F$ is called an atom if for every $V$, $V \leq U$ implies that $V$ is equal to either $0$ or $U$.

Denote by $H_F \subseteq \text{End}(F)$ the set of all atoms in $\text{End}(F)$. 
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Denote by $H_F \subseteq \text{End}(F)$ the set of all atoms in $\text{End}(F)$.

**Some relations on $H_F$**

\[
\begin{align*}
H_F \times (H_F \times H_F) & \supseteq \Delta_F = \{(U, (V, W)) \mid U \leq W \circ V\} \\
(H_F \times H_F) \times H_F & \supseteq \mu_F = \{((U, V), W) \mid U \otimes V \leq W\} \\
H_F \times I & \supseteq \epsilon_F = \{(U, \ast) \mid U \leq \text{id}_F\} \\
I \times H_F & \supseteq \eta_F = \{(*, U) \mid \forall V, W. \ V \otimes U \leq W \Rightarrow V \leq W\} \\
H_F \times H_F & \supseteq S_F = \{(U, V) \mid U \leq \tilde{V}\}
\end{align*}
\]
Reconstruction problem

This structure gives a reconstruction of Hopf algebras in \textbf{Rel}.

\textbf{Theorem (Reconstruction theorem)}

If \( F : \mathbb{C} \rightarrow \text{Rel} \) is \( F_H : \text{Rep}(H) \rightarrow \text{Rel} \) for some \( H \in \text{Rel} \), then there is a canonical isomorphism of Hopf algebras:

\[ H \cong H_{F_H} \]

We describe a sketch of the proof.
Reconstruction problem

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**Theorem (Reconstruction theorem)**

If $F : \mathbf{C} \to \textbf{Rel}$ is $F_H : \text{Rep}(H) \to \textbf{Rel}$ for some $H \in \textbf{Rel}$, then there is a canonical isomorphism of Hopf algebras:

$$H \simeq H_{F_H}$$

We describe a sketch of the proof.

**Notation**

Let $H$ be a Hopf algebra in \textbf{Rel} and $\tau = (X \to X \otimes H) \in \text{Rep}(H)$.

$$x \xrightarrow{a} x' \iff (x, (x', a)) \in \tau$$

**Remark** : $\tau \subseteq X \times (X \times H)$. 
Sketch of the proof.

**Lemma. 1 (Comultiplication)**

For every $a, b, c \in H$, we have:

$$(a, (b, c)) \in \Delta \iff \begin{cases} \forall \tau = (X \to X \otimes H) \in \text{Rep}(H), \\ x \xrightarrow{a} x' \Rightarrow \exists x''. x \xrightarrow{b} x'' \xrightarrow{c} x' \end{cases}$$

**Remark**: $\Delta \subseteq H \times (H \times H)$.

**Lemma. 2 (Multiplication)**

For every $a, b, c \in H$, we have:

$$((a, b), c) \in \mu \iff \begin{cases} \forall \tau = (X \to X \otimes H), \forall \rho = (Y \to Y \otimes H) \in \text{Rep}(H), \\ x \xrightarrow{a} x' \text{ in } \tau \land y \xrightarrow{b} y' \text{ in } \rho \\ \Rightarrow x \otimes y \xrightarrow{c} x' \otimes y' \text{ in } \tau \otimes \rho \end{cases}$$

**Remark**: The underlying set of $\tau \otimes \rho$ is given by $X \times Y = X \otimes Y$. We denote $((x, y)) \in X \otimes Y$ by $x \otimes y$. 
Sketch of the proof

Lemma. 3 (Antipode)

For every $a, b \in H$, we have:

$$(a, b) \in S \iff \forall \tau = (X \to X \otimes H) \in \text{Rep}(H),
\begin{array}{c}
x \xrightarrow{a} x' \text{ in } \tau \implies x' \xrightarrow{b} x \text{ in } \tau^* \end{array}$$

Remark : The underlying set of $\tau^*$ is also $X(=X^*)$ for $\tau = (X \to X \otimes H)$. 
Lemma. 3 (Antipode)
For every \( a, b \in H \), we have:
\[
(a, b) \in S \iff \begin{cases}
\forall \tau = (X \to X \otimes H) \in \text{Rep}(H), \\
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\end{cases}
\]

Remark : The underlying set of \( \tau^* \) is also \( X(= X^*) \) for \( \tau = (X \to X \otimes H) \).
We restate these lemmas in terms of natural transformations. To do so, we need the following notation.

Notation
For \( a \in H \), a natural transformation \( U_a : F \Rightarrow F \) is defined: for each \( \tau = (X \to X \otimes H) \), the component \( U_a(\tau) \subseteq X \times X \) is given by,
\[
U_a(\tau) = \{(x, x') \mid x \xrightarrow{a} x' \text{ in } \tau\}
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Proposition. 1 (Comultiplication)

For every $a, b, c \in H$, we have:

$$(a, (b, c)) \in \Delta \iff U_a \leq U_c \circ U_b$$
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Proposition. 2 (Multiplication)

For every $a, b, c \in H$:

$$((a, b), c) \in \mu \iff U_a \otimes U_b \leq U_c$$

Proposition. 3 (Antipode)

For every $a, b \in H$:

$$(a, b) \in S \iff U_a \leq \bar{U}_b$$
Sketch of the proof

In the case of compact group $G$. [Joyal-Street]

A natural transformation $U : F \Rightarrow F$ on $F : \text{Rep}_f(G, \mathbb{C}) \to \text{Vect}_k$ is of the form $\pi(x)$ for some $x \in G$ if and only if $U$ is self-conjugate and tensor-preserving.
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The notion of atoms characterizes $U_a$.

Proposition. 4

A natural transformation $U : F_H \Rightarrow F_H$ on $F_H : \text{Rep}(H) \to \text{Rel}$ is of the form $U_a$ for some $a \in H$ if and only if $U$ is an atom in $\text{End}(F_H)$.
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A natural transformation $U : F_H \Rightarrow F_H$ on $F_H : \text{Rep}(H) \to \text{Rel}$ is of the form $U_a$ for some $a \in H$ if and only if $U$ is an atom in $\text{End}(F_H)$.

Thus now we can describe the canonical isomorphism from $H$ to $H_{F_H}$:

Canonical isomorphism

The canonical isomorphism is explicitly given by the following correspondence:

$$U_\bullet : H \ni a \mapsto U_a \in H_{F_H}$$
Example (canonical embedding $\text{Sets} \to \text{Rel}$)

Let $F_0 : \text{Sets} \to \text{Rel}$ be the canonical embedding, then the poset $\text{End}(F_0)$ is isomorphic to the poset represented by the following Hasse diagram:

$$
\begin{aligned}
\bullet & \quad \text{id}_{F_0} \\
\circ & \quad 0
\end{aligned}
$$

Thus $H_{F_0}$ is a singleton $\{\text{id}_{F_0}\}$. 

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Some consequences for original problem

We do not forget the original problem.

Original problem

How many monoidal structures can exist on $\text{Aut}(\Sigma)$? Are they finite or infinite? Can we give a good classification of them?

Rough description of $\text{Aut}(\Sigma)$

Objects:

- $a, b$
- $a, b$
- ... non-deterministic automata.

Arrows: (Backward-forward) simulations.
Some consequences for original problem

Typical monoidal structures on $\text{Aut}(\Sigma)$

- CCS-like parallel composition of automata.
- CSP-like parallel composition of automata.
- Interleaving composition of automata.
Some consequences for original problem

**Typical monoidal structures on** $\textbf{Aut}(\Sigma)$

- CCS-like parallel composition of automata.
- CSP-like parallel composition of automata.
- Interleaving composition of automata.

There is a functor $F : \textbf{Aut}(\Sigma) \rightarrow \textbf{Rel}$ that sends an automaton to its state-set, and a simulation to itself. **These typical monoidal structures make** $F : \textbf{Aut}(\Sigma) \rightarrow \textbf{Rel}$ **strict monoidal.**
Typical monoidal structures on $\text{Aut}(\Sigma)$

- CCS-like parallel composition of automata.
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There is a functor $F : \text{Aut}(\Sigma) \rightarrow \text{Rel}$ that sends an automaton to its state-set, and a simulation to itself. These typical monoidal structures make $F : \text{Aut}(\Sigma) \rightarrow \text{Rel}$ strict monoidal.

Restricted classification problem

Classify monoidal structures on $\text{Aut}(\Sigma)$ such that $F : \text{Aut}(\Sigma) \rightarrow \text{Rel}$ is strict monoidal.

**Remark**: In what follows, “monoidal structure” means such monoidal structures.
Some consequences for original problem

Classification of monoidal structures

There is a bijective correspondence between monoidal structures on $\text{Aut}(\Sigma)$ and bialgebra structures on the coalgebra $\Sigma^*$ consisting of finite words.

Example (Interleaving v.s. word shuffling)

The interleaving composition on $\text{Aut}(\Sigma)$ is in correspondence with the shuffling operation on finite words under the above bijective correspondence.

Corollary: $\text{Aut}(\Sigma)$ has only finitely many monoidal structures.

If the set $\Sigma$ consists of $n$ members, then the number $M(n)$ of monoidal structures on $\text{Aut}(\Sigma)$ is finite: there is a rough estimation, $n! \leq M(n) \leq 2^{n^3}$.
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If the set $\Sigma$ consists of $n$ members, then the number $M(n)$ of monoidal structures on $\text{Aut}(\Sigma)$ is finite: there is a rough estimation,

$$n! \leq M(n) \leq 2^{n^3+n}.$$
One can prove the following fact by combinatorial argument on finite words.

**Lemma:** $\Sigma^*$ can not be a Hopf algebra in $\text{Rel}$. 

The coalgebra $\Sigma^*$ can not be a Hopf algebra with respect to any bialgebra structure on it.
Some consequences for original problem

One can prove the following fact by combinatorial argument on finite words.

**Lemma:** $\Sigma^*$ can not be a Hopf algebra in $\text{Rel}$.

The coalgebra $\Sigma^*$ can not be a Hopf algebra with respect to any bialgebra structure on it.

This fact is translated to a fact about $\text{Aut}(\Sigma)$ via Tannaka dualtiy.

**Corollary:** $\text{Aut}(\Sigma)$ cannot be autonomous.

More strongly: for any monoidal structure on $\text{Aut}(\Sigma)$, there exists an automaton that does not have its left dual.
Automata are representations of finite words.

For $F : \text{Aut}(\Sigma) \to \text{Rel}$, we have an equivalence: $\text{Aut}(\Sigma) \simeq \text{Rep}(H_F)$:

- $H_F = \Sigma^*$: the set of finite words.
- $\Delta_F = \{(u, (v, w)) | u = v \cdot w\} \subseteq H_F \times (H_F \times H_F)$
Some consequences for original problem

Automata are representations of finite words.

For $F : \text{Aut}(\Sigma) \rightarrow \text{Rel}$, we have an equivalence: $\text{Aut}(\Sigma) \simeq \text{Rep}(H_F)$:

- $H_F = \Sigma^*$: the set of finite words.
- $\Delta_F = \{(u, (v, w)) \mid u = v \cdot w\} \subseteq H_F \times (H_F \times H_F)$

Example: Automata with permutable paths

$C \subseteq \text{Aut}(\Sigma)$: the full subcategory consisting of automata such that for each $\sigma \in \mathcal{G}_n$,

For the restriction $F : C \rightarrow \text{Rel}$, we have an equivalence $C \simeq \text{Rep}(H_F)$.

- $H_F$: the set of multisets
- $\Delta_F = \{(p, (q, r)) \mid p = q + r\} \subseteq H_F \times (H_F \times H_F)$
4. Some Conjectures
Some Conjectures

Observation

In the reconstruction procedure of $H \in \text{Rel}$, the poset structure of $\text{End}(F)$ plays a key role...why?
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Observation

- $\text{Rel}$ can be embedded into the category $\text{SLat}$
- $\otimes$ on $\text{Rel}$ can be extended to $\otimes$ on $\text{SLat}$.
- $\text{SLat}$ is complete and cocomplete.
Lesson from these observation

The place we should work in is not \textbf{Rel}, but \textbf{SLat} (or something like that).
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Correspondence
- The category $\text{Vect}_k^f$ of finite dimensional spaces is replaced by Rel.
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- The category $\text{Vect}_{k}^{f}$ of finite dimensional spaces is replaced by Rel.
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Conjecture: Fundamental theorem in SLat

For a coalgebra $C \in \text{SLat}$:

$$C = \int_{\tau \in \text{Rep}_{f}(C)} F(\tau)^{*} \otimes F(\tau)$$

where $\text{Rep}_{f}(C)$ consists of representations of $C$ whose underlying set is in Rel, and $F : \text{Rep}_{f}(C) \rightarrow \text{SLat}$ denotes the forgetful functor.
Some Conjectures

Significant point of Tannaka duality

Starting from a category $\mathbf{C}$ which seemingly has nothing to do with coalgebras, one can prove an equivalence of $\mathbf{C}$ and the category $\text{Rep}_f(\mathbf{C})$ of some coalgebra $\mathbf{C}$. 

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Some Conjectures

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Conjecture (hope)

There is a category $\text{Game}$ of some kind of games and a functor $F : \text{Game} \to \text{SLat}$ with $F(\tau)$ in $\text{Rel}$, such that $\text{Game} \simeq \text{Rep}_f(C_F)$. 

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Thank you!