A Note on Bainbridge's Power Set Construction

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Abstract

The category **Rel** of sets and relations has two natural traced monoidal structures: in $(\mathbf{Rel}, +, \mathrm{Tr})$, the tensor is given by disjoint union, and in $(\mathbf{Rel}, \times, \mathrm{Tr}')$ by products of sets. Already in 1976, predating the definition of traced monoidal categories by 20 years, Bainbridge has shown how to model flowcharts and networks in these two respective settings. Bainbridge has also pointed out that one can move from one setting to the other via the power set operation. However, Bainbridge's power operation is not functorial, and in this paper we show that there is no traced monoidal embedding of $(\mathbf{Rel}, +, \mathrm{Tr})$ into $(\mathbf{Rel}, \times, \mathrm{Tr}')$ whose object part is given by the power set operation. On the other hand, we show that there is such an embedding whose object part is given by the power-multiset operation.

Introduction

Predating the definition of traced monoidal categories [2] by 20 years, Bainbridge [1] has pointed out in 1976 that there exist (in today's terminology) two natural traced monoidal structures on the category **Rel** of sets and relations. The first one is $(\mathbf{Rel}, +, \mathrm{Tr})$, where the tensor product is given by disjoint union of sets. The second one is $(\mathbf{Rel}, \times, \mathrm{Tr}')$, where tensor is given by products of sets. Bainbridge used these categories to give a compositional semantics to flowcharts and networks, respectively, and he pointed out a duality between the two situations: the power set operation takes the first category to the second, and it gives rise to a homset-wise Galois connection. Bainbridge's power operation maps a set X to the power set PX, and a relation $R : X \to Y$ to the relation $PR : PX \to PY$ given by $\alpha\ PR\ \beta$ iff for all $x \in \alpha$, xRy implies $y \in \beta$. Remarkably, this operation preserves not only composition and tensor, but also trace. However, it does not preserve identities, and it is therefore not a functor.

One may now ask whether there is some variant of Bainbridge's construction that yields an actual functor of traced monoidal categories. More precisely: is there a traced monoidal embedding of $(\mathbf{Rel}, +, \mathrm{Tr})$ into $(\mathbf{Rel}, \times, \mathrm{Tr}')$ whose object part is given by power sets? The answer, as we shall see, is no. In fact, there is no traced monoidal embedding between these categories that maps finite sets to finite sets. On the other hand, we will show that such an embedding exists whose object part is given by the power-multiset operation.

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An embedding of $(\mathbf{Rel}, +, \mathrm{Tr})$ into $(\mathbf{Rel}, \times, \mathrm{Tr}')$

Let **Rel** be the category of sets and relations, and let $\mathbf{Rel}_{\mathit{fin}}$ be the full subcategory of finite sets. On **Rel**, we consider two traced monoidal structures $(\mathbf{Rel}, +, \mathrm{Tr})$ and $(\mathbf{Rel}, \times, \mathrm{Tr}')$. For the first one, $+$ is disjoint union of sets, and for $R : X + Z \to Y + Z$, $\mathrm{Tr}_Z R : X \to Y$ is given by $x(\mathrm{Tr}_Z R)y$ iff there exist $z_1, \ldots, z_n \in Z$, with $n \geq 0$, such that $xRz_1R\ldots Rz_nRy$. The second traced monoidal structure is given by $\times$ as the product of sets, and for $R : X \times Z \to Y \times Z$, $\mathrm{Tr}'_Z R : X \to Y$ is given by $x(\mathrm{Tr}'_Z R)y$ iff there exists $z \in Z$ such that $(x, z)R(y, z)$. Both these traced monoidal structures restrict to $\mathbf{Rel}_{\mathit{fin}}$. The goal of this section is to prove:

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Theorem 1 *There exists an embedding $F : (\mathbf{Rel}, +, \mathrm{Tr}) \to (\mathbf{Rel}, \times, \mathrm{Tr}')$ of traced monoidal categories.*

Let $N = \{0, 1, \ldots\}$ be the set of natural numbers with addition. For any set X, let $[X \to N]_{\mathit{fin}}$ denote the set of finitely supported X-tuples of natural numbers, i.e. the set of X-tuples $(a_x)_{x\in X}$ such that for all but finitely many $x \in X$, $a_x = 0$ (notice that these tuples could be regarded as finite multisets). If $(a_x)_x$, $(b_y)_y$, and $(e_{xy})_{xy}$ are such tuples, then we write

$$\begin{array}{c|ccc} & b_y & \cdots & b_{y'} \\ \hline a_x & e_{xy} & \cdots & e_{xy'} \\ \vdots & \vdots & \ddots & \vdots \\ a_{x'} & e_{x'y} & \cdots & e_{x'y'} \end{array}$$

as a suggestive notation for

$$\begin{array}{ll} a_x = \sum_{y\in Y} e_{xy} & \text{for all } x \in X \text{ and} \\ b_y = \sum_{x\in X} e_{xy} & \text{for all } y \in Y. \end{array}$$

We use this notation for infinite as well as for finite index sets, which is justified since the tuples are finitely supported.

Lemma 2 *There exist $(e_{xy})_{xy}$ satisfying the above equations if and only if $\sum_{x\in X} a_x = \sum_{y\in Y} b_y$.*

Proof: The "only if" direction is trivial, the other direction follows by induction on $\sum_{x\in X} a_x$. □

We now construct a functor $F : \mathbf{Rel} \to \mathbf{Rel}$ as follows. For any set X, let $FX = [X \to N]_{\mathit{fin}}$. On morphisms $R : X \to Y$, we define $FR : FX \to FY$ to be the relation given by $(a_x)_x \; FR \; (b_y)_y$ if and only if there exist $(e_{xy})_{xy}$ such that $e_{xy} \neq 0$ implies xRy for all x, y, and such that

$$\begin{array}{c|ccc} & b_y & \cdots & b_{y'} \\ \hline a_x & e_{xy} & \cdots & e_{xy'} \\ \vdots & \vdots & \ddots & \vdots \\ a_{x'} & e_{x'y} & \cdots & e_{x'y'} \end{array}$$

It is easy to see that if $R : X \to X$ is the identity relation, then $(a_x)_x \; FR \; (b_x)_x$ iff for all x, $a_x = b_x$. Thus, F preserves identities. To see that F preserves composition[1], consider $R : X \to Y$ and $S : Y \to Z$. Suppose $(a_x)_x \; FR \; (b_y)_y \; FS \; (c_z)_z$ via

$$\begin{array}{c|ccc} & b_y & \cdots & b_{y'} \\ \hline a_x & e_{xy} & \cdots & e_{xy'} \\ \vdots & \vdots & \ddots & \vdots \\ a_{x'} & e_{x'y} & \cdots & e_{x'y'} \end{array} \qquad \text{and} \qquad \begin{array}{c|ccc} & c_z & \cdots & c_{z'} \\ \hline b_y & f_{yz} & \cdots & f_{yz'} \\ \vdots & \vdots & \ddots & \vdots \\ b_{y'} & f_{y'z} & \cdots & f_{y'z'} \end{array}$$

such that $e_{xy} \neq 0$ implies xRy, and $f_{yz} \neq 0$ implies ySz. By Lemma 2, for every $y \in Y$ there is $(g_{xyz})_{xz}$ such that

$$\begin{array}{c|ccc} & f_{yz} & \cdots & f_{yz'} \\ \hline e_{xy} & g_{xyz} & \cdots & g_{xyz'} \\ \vdots & \vdots & \ddots & \vdots \\ e_{x'y} & g_{x'yz} & \cdots & g_{x'yz'} \end{array}$$

Let $h_x z = \sum_y g_{xyz}$. Then for all $x \in X$,

$$a_x = \sum_y e_{xy} = \sum_{y,z} g_{xyz} = \sum_z h_{xz},$$

[1] Since in any traced monoidal category, $f; g = \mathrm{Tr}((f \otimes g); c)$, it would suffice to check this for the case where $g = c$.

and similarly $c_z = \sum_x h_{xz}$ for all $z \in Z$. Thus

$$\begin{array}{c|ccc} & c_z & \cdots & c_{z'} \\ \hline a_x & h_{xz} & \cdots & h_{xz'} \\ \vdots & \vdots & \ddots & \vdots \\ a_{x'} & h_{x'z} & \cdots & h_{x'z'} \end{array}$$

Moreover, if $h_{xz} \neq 0$ then there exists y such that $g_{xyz} \neq 0$, hence $e_{xy} \neq 0$ and $f_{yz} \neq 0$, hence xRy and ySz, hence $xRSz$. Thus, $(a_x)_x\ F(RS)\ (c_z)_z$. This shows that $(FR)(FS) \subseteq F(RS)$.

Conversely, assume that $(a_x)_x\ F(RS)\ (c_z)_z$ via

$$\begin{array}{c|ccc} & c_z & \cdots & c_{z'} \\ \hline a_x & h_{xz} & \cdots & h_{xz'} \\ \vdots & \vdots & \ddots & \vdots \\ a_{x'} & h_{x'z} & \cdots & h_{x'z'} \end{array}$$

such that $h_{xz} \neq 0$ implies $xRSz$. For each pair (x, z) such that $xRSz$, choose a particular $y_{xz} \in Y$ such that $xRy_{xz}Sz$. Define

$$\begin{aligned} g_{xyz} &= \begin{cases} h_{xz} & \text{if } y = y_{xz}, \\ 0 & \text{else,} \end{cases} \\ b_y &= \sum_{x,z} g_{xyz}, \\ e_{xy} &= \sum_z g_{xyz}, \\ f_{yz} &= \sum_x g_{xyz}. \end{aligned}$$

Then

$$\begin{aligned} \sum_y g_{xyz} &= h_{xz}, \\ \sum_x e_{xy} &= \sum_{x,z} g_{xyz} = b_y, \\ \sum_y e_{xy} &= \sum_{y,z} g_{xyz} = \sum_z h_{xz} = a_x, \\ \sum_y f_{yz} &= \sum_{x,y} g_{xyz} = \sum_x h_{xz} = c_z, \\ \sum_z f_{yz} &= \sum_{x,z} g_{xyz} = b_y, \end{aligned}$$

thus

$$\begin{array}{c|ccc} & b_y & \cdots & b_{y'} \\ \hline a_x & e_{xy} & \cdots & e_{xy'} \\ \vdots & \vdots & \ddots & \vdots \\ a_{x'} & e_{x'y} & \cdots & e_{x'y'} \end{array} \quad \text{and} \quad \begin{array}{c|ccc} & c_z & \cdots & c_{z'} \\ \hline b_y & f_{yz} & \cdots & f_{yz'} \\ \vdots & \vdots & \ddots & \vdots \\ b_{y'} & f_{y'z} & \cdots & f_{y'z'} \end{array}$$

Moreover, if $e_{xy} \neq 0$, then for some z, $g_{xyz} \neq 0$, hence $y = y_{xz}$, hence xRy. Similarly, if $f_{yz} \neq 0$, then ySz. It follows that $(a_x)_x\ FR\ (b_y)_y\ FS\ (c_z)_z$, and thus $F(RS) \subseteq (FR)(FS)$. We have shown that F is a functor.

Next, we show that F preserves the symmetric monoidal structure. On objects, $F(X+Y)\cong F(X)\times F(Y)$ via the identification of $(a_i)_{i\in X+Z}$ with $((a_x)_{x\in X},(a_z)_{z\in Z})$. Moreover, $F(0)\cong 1$. For morphisms, consider $R:X\to Y$ and $S:Z\to W$. Then $R+S:X+Z\to Y+W$. Assume $((a_x)_x,(c_z)_z)\ F(R+S)\ ((b_y)_y,(d_w)_w)$ via

$$\begin{array}{c|cccccc}
 & b_y & \cdots & b_{y'} & d_w & \cdots & d_{w'} \\ \hline
a_x & e_{xy} & \cdots & e_{xy'} & e_{xw} & \cdots & e_{xw'} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
a_{x'} & e_{x'y} & \cdots & e_{x'y'} & e_{x'w} & \cdots & e_{x'w'} \\
c_z & e_{zy} & \cdots & e_{zy'} & e_{zw} & \cdots & e_{zw'} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
c_{z'} & e_{z'y} & \cdots & e_{z'y'} & e_{z'w} & \cdots & e_{z'w'}
\end{array}$$

where $e_{ij}\neq 0$ implies $i(R+S)j$. Thus, in particular, $e_{xw}=0$ for all $x\in X$ and $w\in W$, and $e_{yz}=0$ for all $y\in Y$ and $z\in Z$. Hence $(a_x)_x\ FR\ (b_y)_y$ and $(c_z)_z\ FR\ (d_w)_w$. The converse is also trivial, and thus we have $F(R+S)=FR\times FS$. Last, F preserves the canonical isomorphisms for associativity, unit, and symmetry.

We will now show that F preserves trace. Consider $R:X+Z\to Y+Z$ and let $Q=\mathrm{Tr}_Z R:X\to Y$. First, suppose that $(a_x)_x\ FQ\ (b_y)_y$ via

$$\begin{array}{c|ccc}
 & b_y & \cdots & b_{y'} \\ \hline
a_x & e_{xy} & \cdots & e_{xy'} \\
\vdots & \vdots & \ddots & \vdots \\
a_{x'} & e_{x'y} & \cdots & e_{x'y'}
\end{array}$$

where $e_{xy}\neq 0$ implies xQy. Now choose a set of words $A\subseteq X\times Z^*\times Y$ such that

1. whenever $xz_1\ldots z_n y\in A$, then $xRz_1R\ldots Rz_nRy$, and
2. for each pair (x,y) with xQy, there is exactly one word $xz_1\ldots z_n y\in A$.

If w and w' are words, then we say w is a subword of w', in symbols $w\lhd w'$, if there exist words u and v such that $uwv=w'$. In the following, we denote words in Z^* by ξ. For $i\in X+Z$ and $j\in Y+Z$, define

$$\begin{aligned}
f_{ij} &= \sum\{e_{xy}\ \mid\ ij\lhd x\xi y\in A\},\\
c_z &= \sum\{e_{xy}\ \mid\ z\lhd x\xi y\in A\}.
\end{aligned}$$

Notice that these sums are finite, because only finitely many $e_{xy}\neq 0$. Then

$$\begin{aligned}
\sum_{j\in Y+Z} f_{xj} &= \sum_{y\ \mid\ x\xi y\in A} e_{xy} = \sum_{y\ \mid\ xQy} e_{xy} = \sum_y e_{xy} = a_x,\\
\sum_{i\in X+Z} f_{iy} &= \sum_{x\ \mid\ x\xi y\in A} e_{xy} = \sum_{x\ \mid\ xQy} e_{xy} = \sum_x e_{xy} = b_y,\\
\sum_{j\in Y+Z} f_{zj} &= \sum_{x,y\ \mid\ z\lhd x\xi y\in A} e_{xy} = c_z,\\
\sum_{i\in X+Z} f_{iz} &= \sum_{x,y\ \mid\ z\lhd x\xi y\in A} e_{xy} = c_z.
\end{aligned}$$

Thus

$$\begin{array}{c|cccccc}
 & b_y & \cdots & b_{y'} & c_z & \cdots & c_{z'} \\ \hline
a_x & f_{xy} & \cdots & f_{xy'} & f_{xz} & \cdots & f_{xz'} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
a_{x'} & f_{x'y} & \cdots & f_{x'y'} & f_{x'z} & \cdots & f_{x'z'} \\
c_z & f_{zy} & \cdots & f_{zy'} & f_{zz} & \cdots & f_{zz'} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
c_{z'} & f_{z'y} & \cdots & f_{z'y'} & f_{z'z} & \cdots & f_{z'z'}
\end{array}$$

Moreover, if $f_{ij} \neq 0$, then there exists $x\xi y \in A$ with $ij \triangleleft x\xi y$, hence iRj by definition of A. Thus, it follows that $((a_x)_x, (c_z)_z)\ FR\ ((b_y)_y, (c_z)_z)$, and therefore $(a_x)_x\ \mathrm{Tr}'_{FZ}(FR)\ (b_y)_y$. This shows $F(\mathrm{Tr}_Z R) \subseteq \mathrm{Tr}'_{FZ}(FR)$.

For the converse, assume that $(a_x)_x\ \mathrm{Tr}'_{FZ}(FR)\ (b_y)_y$ holds. By definition of Tr', there exists $(c_z)_z$ such that $((a_x)_x, (c_z)_z)\ FR\ ((b_y)_y, (c_z)_z)$. Let $(f_{ij})_{i\in X+Z, j\in Y+Z}$ be such that $f_{ij} \neq 0$ implies iRj and

$$\begin{array}{c|cccccc}
 & b_y & \cdots & b_{y'} & c_z & \cdots & c_{z'} \\ \hline
a_x & f_{xy} & \cdots & f_{xy'} & f_{xz} & \cdots & f_{xz'} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
a_{x'} & f_{x'y} & \cdots & f_{x'y'} & f_{x'z} & \cdots & f_{x'z'} \\
c_z & f_{zy} & \cdots & f_{zy'} & f_{zz} & \cdots & f_{zz'} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
c_{z'} & f_{z'y} & \cdots & f_{z'y'} & f_{z'z} & \cdots & f_{z'z'}
\end{array}$$

Again, let $Q = \mathrm{Tr}_Z R$. We will show that $(a_x)_x\ FQ\ (b_y)_y$ by induction on $\sum_z c_z$. We distinguish three cases:

- Case 1: $\sum_z c_z = 0$. Then for all i, j, z, $f_{iz} = 0$ and $f_{zj} = 0$, hence

$$\begin{array}{c|ccc}
 & b_y & \cdots & b_{y'} \\ \hline
a_x & f_{xy} & \cdots & f_{xy'} \\
\vdots & \vdots & \ddots & \vdots \\
a_{x'} & f_{x'y} & \cdots & f_{x'y'}
\end{array}$$

  and $f_{xy} \neq 0$ implies xRy implies xQy, hence we are done.

- Case 2: There exists $n \geq 2$ and distinct $z_1, \ldots, z_n \in Z$ such that $f_{z_1 z_2}, \ldots, f_{z_{n-1} z_n}, f_{z_n, z_1} \neq 0$. Define

$$c'_z = \begin{cases} c_z - 1 & \text{if } z \in z_1, \ldots, z_n, \\ c_z & \text{else,} \end{cases}$$
$$f'_{ij} = \begin{cases} f_{ij} - 1 & \text{if } ij \triangleleft z_1 \ldots z_n z_1, \\ f_{ij} & \text{else.} \end{cases}$$

  Then

$$\begin{array}{c|cccccc}
 & b_y & \cdots & b_{y'} & c'_z & \cdots & c'_{z'} \\ \hline
a_x & f'_{xy} & \cdots & f'_{xy'} & f'_{xz} & \cdots & f'_{xz'} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
a_{x'} & f'_{x'y} & \cdots & f'_{x'y'} & f'_{x'z} & \cdots & f'_{x'z'} \\
c'_z & f'_{zy} & \cdots & f'_{zy'} & f'_{zz} & \cdots & f'_{zz'} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
c'_{z'} & f'_{z'y} & \cdots & f'_{z'y'} & f'_{z'z} & \cdots & f'_{z'z'}
\end{array}$$

  holds and $f'_{ij} \neq 0$ still implies iRj, moreover $\sum_z c'_z < \sum_z c_z$. By induction hypothesis, we get $(a_x)_x\ FQ\ (b_y)_y$.

- Case 3: Since we are not in Case 1, we can assume that there is $i_0 \in Z$ such that $c_{i_0} \neq 0$. Inductively suppose that we are given $i_k \in Z$ such that $c_{i_k} \neq 0$, then there is $i_{k+1} \in X + Z$ such that $f_{i_{k+1} i_k} \neq 0$, and thus also $c_{i_{k+1}} \neq 0$. In this manner, we construct a sequence $i_0, i_1, i_2, \ldots$. Since we are not in Case 2, this sequence is non-repeating, and since only finitely many c_z are different from 0, the sequence must eventually stop with some $i_k \in X$. Proceeding from i_0 in the other direction, we can construct a similar sequence, so that in the end we get a path $\hat{x}z_0 \dots z_n\hat{y}$ with $f_{\hat{x}z_0}, f_{z_0z_1}, \dots, f_{z_n\hat{y}} \neq 0$. Define

$$a'_x = \begin{cases} a_x - 1 & \text{if } x = \hat{x}, \\ a_x & \text{else,} \end{cases}$$

$$b'_y = \begin{cases} b_y - 1 & \text{if } y = \hat{y}, \\ b_y & \text{else,} \end{cases}$$

$$c'_z = \begin{cases} c_z - 1 & \text{if } z \in z_0, \dots, z_n, \\ c_z & \text{else,} \end{cases}$$

$$f'_{ij} = \begin{cases} f_{ij} - 1 & \text{if } ij \lhd \hat{x}z_0 \dots z_n\hat{y}, \\ f_{ij} & \text{else.} \end{cases}$$

Then

$$\begin{array}{c|cccccc} & b'_y & \cdots & b'_{y'} & c'_z & \cdots & c'_{z'} \\ \hline a'_x & f'_{xy} & \cdots & f'_{xy'} & f'_{xz} & \cdots & f'_{xz'} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a'_{x'} & f'_{x'y} & \cdots & f'_{x'y'} & f'_{x'z} & \cdots & f'_{x'z'} \\ c'_z & f'_{zy} & \cdots & f'_{zy'} & f'_{zz} & \cdots & f'_{zz'} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ c'_{z'} & f'_{z'y} & \cdots & f'_{z'y'} & f'_{z'z} & \cdots & f'_{z'z'} \end{array}$$

and $f'_{ij} \neq 0$ still implies iRj; moreover $\sum_z c'_z < \sum_z c_z$. By induction hypothesis, we get $(a'_x)_x \ FQ \ (b'_y)_y$ via some

$$\begin{array}{c|ccc} & b'_y & \cdots & b'_{y'} \\ \hline a'_x & e'_{xy} & \cdots & e'_{xy'} \\ \vdots & \vdots & \ddots & \vdots \\ a'_{x'} & e'_{x'y} & \cdots & e'_{x'y'} \end{array}$$

where $e'xy \neq 0$ implies xQy. Now let

$$e_{xy} = \begin{cases} e'_{xy} + 1 & \text{if } x = \hat{x} \text{ and } y = \hat{y}, \\ e'_{xy} & \text{else.} \end{cases}$$

Then

$$\begin{array}{c|ccc} & b_y & \cdots & b_{y'} \\ \hline a_x & e_{xy} & \cdots & e_{xy'} \\ \vdots & \vdots & \ddots & \vdots \\ a_{x'} & e_{x'y} & \cdots & e_{x'y'} \end{array}$$

and if $e_{xy} \neq 0$, then either $e'_{xy} \neq 0$, in which case xQy, or else $x = \hat{x}$ and $y = \hat{y}$. But, by construction, $\hat{x}Rz_0R \dots Rz_nR\hat{y}$, and thus xQy. Thus $(a_x)_x \ FQ \ (b_y)_y$, which shows that $\mathrm{Tr}'_{FZ}(FR) \subseteq F(\mathrm{Tr}_Z R)$, thereby finishing the proof of Theorem 1. $\square$

There is no embedding of $(\mathbf{Rel}_{\mathit{fin}}, +, \mathrm{Tr})$ into $(\mathbf{Rel}_{\mathit{fin}}, \times, \mathrm{Tr}')$

In this section, we will show that Bainbridge's construction cannot be made into a functor from $(\mathbf{Rel}, +, \mathrm{Tr})$ into $(\mathbf{Rel}, \times, \mathrm{Tr}')$. More generally:

Theorem 3 *There exists no embedding $F : (\mathbf{Rel}_{\mathit{fin}}, +, \mathrm{Tr}) \to (\mathbf{Rel}_{\mathit{fin}}, \times, \mathrm{Tr}')$ of traced monoidal categories.*

Notice that this theorem implies that there is no embedding of $(\mathbf{Rel}, +, \mathrm{Tr})$ into $(\mathbf{Rel}, \times, \mathrm{Tr}')$ which is given by the power set operation on objects.

For any finite set N, let $F_N : (\mathbf{Rel}_{\mathit{fin}}, +) \to (\mathbf{Rel}_{\mathit{fin}}, \times)$ be the symmetric monoidal functor that is given on objects by $F_N X = N^X$ and on morphisms by $(a_x)_{x\in X}\ F_N R\ (b_y)_{y\in Y}$ iff for all x, y, xRy implies $a_x = b_y$. One checks that the functor F_N preserves trace if and only if $N \neq \emptyset$. However, F_N is never an embedding.

For an arbitrary symmetric monoidal functor $F : (\mathbf{Rel}_{\mathit{fin}}, +) \to (\mathbf{Rel}_{\mathit{fin}}, \times)$, we will show that if F preserves trace, then it is naturally isomorphic to F_N for some N. In particular, there is no traced monoidal embedding $F : (\mathbf{Rel}_{\mathit{fin}}, +, \mathrm{Tr}) \to (\mathbf{Rel}_{\mathit{fin}}, \times, \mathrm{Tr}')$.

Given such a functor F, let $N = F(1)$. Notice that any object X in $\mathbf{Rel}_{\mathit{fin}}$ is of the form $X = 1 + 1 + \ldots + 1$, and thus, $FX = N \times N \times \ldots \times N = N^X$. For any two objects X and Y, let $\nabla_{XY} \colon N^X \to N^Y = F(\top_{XY})$ be the image of the full relation $\top_{XY} : X \to Y$, i.e. of the relation $\top_{XY} = X \times Y$.

Notice that F is completely determined (up to natural isomorphism) by N and the relations ∇_{XY}, because any morphism $R : X \to Y$ in **Rel** can be written as

$$X = \sum_{x\in X} 1 \xrightarrow{\sum_x \top_{1Y}} \sum_{x\in X} Y \cong \sum_{x\in X, y\in Y} 1 \xrightarrow{\sum_{xy} R_{xy}} \sum_{x\in X, y\in Y} 1 \cong \sum_{y\in Y} X \xrightarrow{\sum_y \top_{X1}} \sum_{y\in Y} 1 = Y,$$

where

$$R_{xy} : 1 \to 1 = \begin{cases} \mathrm{id}_1 & \text{if } xRy, \\ 1 \xrightarrow{\top_{10}} 0 \xrightarrow{\top_{01}} 1 & \text{else.} \end{cases}$$

Thus, FR can be computed from ∇_{XY} via

$$N^X = \prod_{x\in X} N \xrightarrow{\prod_x \nabla_{1Y}} \prod_{x\in X} N^Y \cong \prod_{x\in X, y\in Y} N \xrightarrow{\prod_{xy} F(R_{xy})} \prod_{x\in X, y\in Y} N \cong \prod_{y\in Y} N^X \xrightarrow{\prod_y \nabla_{X1}} \prod_{y\in Y} N = Y,$$

where

$$F(R_{xy}) : N \to N = \begin{cases} \mathrm{id}_N & \text{if } xRy, \\ N \xrightarrow{\nabla_{10}} 1 \xrightarrow{\nabla_{01}} N & \text{else.} \end{cases}$$

Let $\bullet$ be a tag such that $\bullet \notin N$. We extend the relations ∇_{XY} to $\overset{\bullet}{\nabla}_{XY} \subseteq (N + \{\bullet\})^X \times (N + \{\bullet\})^Y$ by setting $(a_x)_x\ \overset{\bullet}{\nabla}_{XY}\ (b_y)_y$ if and only if there exist $(a'_x)_x$ and $(b'_y)_y$ such that $(a'_x)_x\ \nabla_{XY}\ (b'_y)_y$ and

$$\begin{array}{ll} a'_x = a_x & \text{if } a_x \neq \bullet, \\ a'_x \in \nabla_{01} & \text{if } a_x = \bullet, \\ b'_y = b_y & \text{if } b_y \neq \bullet, \\ b'_y \in \nabla_{10} & \text{if } b_y = \bullet. \end{array}$$

If $(a_x)_{x\in X}$ and $(b_y)_{y\in Y}$ are tuples in N, and $(e_{xy})_{x\in X, y\in Y}$ is a tuple in $N + \{\bullet\}$, we will write

$$\begin{array}{c|ccc} & b_y & \cdots & b_{y'} \\ \hline a_x & e_{xy} & \cdots & e_{xy'} \\ \vdots & \vdots & \ddots & \vdots \\ a_{x'} & e_{x'y} & \cdots & e_{x'y'} \end{array}$$

as an abbreviation for

$$\begin{array}{rccl} a_x & \overset{\bullet}{\nabla}_{1Y} & (e_{xy})_y & \text{for all } x \in X \text{ and} \\ (e_{xy})_x & \overset{\bullet}{\nabla}_{X1} & b_y & \text{for all } y \in Y, \end{array}$$

Lemma 4 *$(a_x)_x\ FR\ (b_y)_y$ if and only if there exist a tuple $(e_{xy})_{x\in X, y\in Y}$ of elements of $N + \{\bullet\}$, such that $e_{xy} \neq \bullet$ iff xRy, and such that*

$$\begin{array}{c|ccc} & b_y & \cdots & b_{y'} \\ \hline a_x & e_{xy} & \cdots & e_{xy'} \\ \vdots & \vdots & \ddots & \vdots \\ a_{x'} & e_{x'y} & \cdots & e_{x'y'} \end{array}$$

Proof: We already know that $(a_x)_x\ FR\ (b_y)_y$ if and only if

$$(a_x)_x\ (\prod_x \nabla_{1Y}; \prod_{xy} F(R_{xy}); \prod_y \nabla_{X1})\ (b_y)_y,$$

which is the case if and only if there exist $(e'_{xy})_{x\in X, y\in Y}$ and $(e'_{xy})_{x\in X, y\in Y}$ from N such that

$$\begin{array}{ll} a_x\ \ \nabla_{1Y}\ \ (e'_{xy})_y & \text{for all } x \in X, \\ e'_{xy} = e''_{xy} & \text{for all } xRy, \\ e'_{xy} \in \nabla_{10} \text{ and } e''_{xy} \in \nabla_{01} & \text{for all } x \not R\ y, \\ (e''_{xy})_x\ \ \nabla_{X1}\ \ b_y & \text{for all } y \in Y. \end{array}$$

Now letting $e_{xy} = e'_{xy} = e''_{xy}$ if xRy, and $e_{xy} = \bullet$ if $x \not R\ y$, the claim follows. $\square$

Lemma 5 *The following statements, along with their duals, are properties of the relations ∇_{XY}:*

1. *For all $(a_x)_x$ and all permutations $\phi : X \to X$, one has $b\ \nabla_{1X}\ (a_x)_x$ iff $b\ \nabla_{1X}\ (a_{\phi x})_x$.*

2. *If $e \in \nabla_{10}$, then $a\ \nabla_{1,X+1}\ (b_1, \ldots, b_X, e)$ implies $a\ \nabla_{1,X}\ (b_1, \ldots, b_X)$. Conversely, whenever $a\ \nabla_{1,X}\ (b_1, \ldots, b_X)$, then there exists an $e \in \nabla_{10}$ such that $a\ \nabla_{1,X+1}\ (b_1, \ldots, b_X, e)$. (Actually, e depends only on a, but we don't need this fact).*

3. *For every $b \in N$, and every $n \geq 1$, there exists $(a_1, \ldots, a_n)$ such that $b\ \nabla_{1n}\ (a_1, \ldots, a_n)\ \nabla_{n1}\ b$.*

4. *For every $b \in N$ and every $n \geq 1$, there exists $a \in N$ such that $(b, \ldots, b)\ \nabla_{n1}\ a\ \nabla_{1n}\ (b, \ldots, b)$.*

5. *If $\nabla_{01} \neq N$, then there exist $a, b \in N$ such that $a\ \nabla_{12}\ (b, a)$ and $b \notin \nabla_{01}$.*

Proof:

1. Consider the following two diagrams. The left diagram communtes in $(\mathbf{Rel}_{\mathit{fin}}, +)$. By applying F, one gets the right diagram in $(\mathbf{Rel}_{\mathit{fin}}, \times)$:

$$\begin{tikzcd} 1 \arrow[r, "\top_{1X}"] \arrow[dr, "\top_{1X}"'] & X \arrow[d, "\phi"] \\ & X \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} N \arrow[r, "\nabla_{1X}"] \arrow[dr, "\nabla_{1X}"'] & N^X \arrow[d, "F\phi"] \\ & N^X. \end{tikzcd}$$

   But since ϕ is given in terms of the symmetric monoidal structure, $F\phi$ behaves as expected, which implies the claim.

2. Again, commutativity of the left diagram implies commutativity of the right one:

$$\begin{tikzcd} 1 \arrow[r, "\top_{1,X+1}"] \arrow[dr, "\top_{1X}"'] & X+1 \arrow[d, "\mathrm{id}+\top_{10}"] \\ & X \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} N \arrow[r, "\nabla_{1,X+1}"] \arrow[dr, "\nabla_{1X}"'] & N^X \times N \arrow[d, "\mathrm{id}\times\nabla_{10}"] \\ & N^X, \end{tikzcd}$$

   Thus, $a\ \nabla_{1,X}\ (b_1, \ldots, b_X)$ iff there exists $e \in \nabla_{01}$ with $a\ \nabla_{1,X+1}\ (b_1, \ldots, b_X, e)$, which was the claim.

3. Again, we transfer a diagram from $(\mathbf{Rel}_{\mathit{fin}}, +)$ to $(\mathbf{Rel}_{\mathit{fin}}, \times)$ along F:

$$\begin{tikzcd} 1 \arrow[r, "\top_{1n}"] \arrow[dr, "\mathrm{id}"'] & n \arrow[d, "\top_{n1}"] \\ & 1 \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} N \arrow[r, "\nabla_{1n}"] \arrow[dr, "\mathrm{id}"'] & N^n \arrow[d, "\nabla_{n1}"] \\ & N. \end{tikzcd}$$

   The claim follows.

4. Suppose $b \in N$ and $n \geq 1$ are given. By (3), there is $(a_1, \ldots, a_n)$ such that $b \nabla_{1n} (a_1, \ldots, a_n) \nabla_{n1} b$. Then

$$
\begin{array}{c|ccccc}
 & b & b & \cdots & b & b \\
\hline
b & a_1 & a_2 & \cdots & a_{n-1} & a_n \\
b & a_n & a_1 & \cdots & a_{n-2} & a_{n-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
b & a_3 & a_4 & \cdots & a_1 & a_2 \\
b & a_2 & a_3 & \cdots & a_n & a_1
\end{array}
$$

and thus $(b, \ldots, b) \nabla_{nn} (b, \ldots, b)$ by Lemma 4. But

$$
\begin{tikzcd}
n \arrow[r, "\top_{n1}"] \arrow[dr, "\top_{nn}"'] & 1 \arrow[d, "\top_{1n}"] \\
 & n
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
N^n \arrow[r, "\nabla_{n1}"] \arrow[dr, "\nabla_{nn}"'] & N \arrow[d, "\nabla_{1n}"] \\
 & N^n,
\end{tikzcd}
$$

and hence there exists $a \in N$ such that $(b, \ldots, b) \nabla_{n1} a \nabla_{1n} (b, \ldots, b)$.

5. Suppose there is some $c \in N$ with $c \notin \nabla_{01}$. For each $n \geq 1$, use (4) to choose $d_n \in N$ such that $(c, \ldots, c) \nabla_{n1} d_n \nabla_{1n} (c, \ldots, c)$. Since N is finite, there must be $n, m \geq 1$ such that $d_n = d_{n+m}$. Now let $a = d_n = d_{n+m}$, and let $b = d_m$. Then
$$a \ \nabla_{1,n+m} \ (c, \overset{n}{\ldots}, c, c, \overset{m}{\ldots}, c) \ (\nabla_{n1} \times \nabla_{m1}) \ (a, b).$$
But we have

$$
\begin{tikzcd}
1 \arrow[r, "\top_{1,n+m}"] \arrow[dr, "\top_{12}"'] & n+m \arrow[d, "\top_{n1}+\top_{m1}"] \\
 & 1+1
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
N \arrow[r, "\nabla_{1,n+m}"] \arrow[dr, "\nabla_{12}"'] & N^n \times N^m \arrow[d, "\nabla_{n1}\times\nabla_{m1}"] \\
 & N \times N,
\end{tikzcd}
$$

and hence it follows that $a \nabla_{12} (a, b)$. Moreover, suppose $b \in \nabla_{01}$. Because $b \nabla_{1m} (c, \overset{m}{\ldots}, c)$, it follows that $(c, \overset{m}{\ldots}, c) \in \nabla_{0m}$. But

$$
\begin{aligned}
\top_{0m} &= 0 \xrightarrow{\top_{01}+\ldots+\top_{01}} m \\
\Rightarrow \quad \nabla_{0m} &= 1 \xrightarrow{\nabla_{01}\times\ldots\times\nabla_{01}} N^m,
\end{aligned}
$$

hence $c \in \nabla_{01}$, a contradiction. $\square$

Up to this point, we have derived properties of an arbitrary symmetric monoidal functor $F : (\mathbf{Rel}_{\mathit{fin}}, +) \to (\mathbf{Rel}_{\mathit{fin}}, \times)$. Notice that the only time we have used the finiteness of N was in the last part of Lemma 5. Now, assume that F preserves trace. We will show that $\nabla_{01} = N$. By way of contradiction, assume that $\nabla_{01} \neq N$. Then, by Lemma 5(5), there exist $a, b \in N$ with $a \nabla_{12} (b, a)$ and $b \notin \nabla_{01}$. Moreover, we can easily find $c, d \in N$ with $(c, d) \nabla_{21} c$, for instance by Lemma 5(2). Now let $X = \{x\}$, $Y = \{y\}$, and $Z = \{z_1, z_2\}$. Consider the relation $R : X + Z \to Y + Z$ given by the matrix
$$R = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
i.e., xRz_2, z_2Rz_2, z_1Rz_1, and z_1Ry. From $a \nabla_{12} (b, a)$ and $(c, d) \nabla_{21} c$, with Lemma 5(2), it follows that

$$
\begin{array}{c|ccc}
 & b & a & c \\
\hline
d & \bullet & \bullet & d \\
a & b & a & \bullet \\
c & \bullet & \bullet & c
\end{array}
$$

and thus, by Lemma 4, $(d, a, c) FR (b, a, c)$. By definition of the trace on $(\mathbf{Rel}_{\mathit{fin}}, \times, \mathrm{Tr}')$, it follows that $d\,\mathrm{Tr}'_{FZ}(FR)\,b$. Since F preserves trace, we must have $d\,F(\mathrm{Tr}_Z R)\,b$. But notice that $\mathrm{Tr}_Z R : X \to Y$ is the empty relation. From $d\,F(\emptyset)\,b$, it follows by Lemma 4 that $b \in \nabla_{01}$, a contradiction. Therefore, it must have been the case that $\nabla_{01}= N$.

By the dual argument, we also have $\nabla_{10}= N$. Now we can apply Lemma 5(2) to arbitrary $e \in N$, and by repeatedly doing so, it follows that for all a and $(b_y)_y$, if $a\ \nabla_{1Y}\ (b_y)_y$, then $a = b_y$ for all $y \in Y$. (Note that $\nabla_{11}=\mathrm{id}_N$). Conversely, if $(b_y)_y$ is a constant tuple, then by Lemma 5(4), there exists a with $a\ \nabla_{1Y}\ (b_y)_y$. Thus

$$a\ \nabla_{1Y}\ (b_y)_y \qquad \text{if and only if} \qquad a = b_y \text{ for all } y \in Y.$$

Similarly, the dual statement holds, and by writing $\nabla_{XY}=\nabla_{X1}\nabla_{1Y}$, we get

$$(a_x)_x\ \nabla_{XY}\ (b_y)_y \qquad \text{if and only if} \qquad a_x = b_y \text{ for all } x \in X, y \in Y.$$

From here, it is easily seen that F is naturally isomorphic to the functor F_N defined at the beginning of this section (recall that F is uniquely determined by N and the relations ∇_{XY}). This concludes the proof of Theorem 3. □

Additional challenges

Notice that the proof of Theorem 3 only uses the trace of one particular matrix, namely

$$R = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Can one extract from this proof a universal sentence (in the predicates of traced monoidal categories and equality) which holds in $(\mathbf{Rel}_{\mathit{fin}}, \times, \mathrm{Tr}')$ but not in $(\mathbf{Rel}_{\mathit{fin}}, +, \mathrm{Tr})$? Such a (possibly infinite) sentence must exist by abstract algebraic nonsense. But a nice such sentence would yield a possibly more elegant proof of the non-embedding theorem.

References

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