Axiomatic description of mixed states from Selinger's CPM-construction

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Abstract

We recast Selinger's CPM-construction of completely positive maps [11] as an axiomatization of maximally mixed states. This axiomatization also guarantees categories of completely positive maps to satisfy the preparation-state agreement axiom of [3], and admits a physical interpretation in terms of purification of mixed states and CPMs. Internal traces, which are crucial in quantum information theory, are the adjoints to these maximally mixed states.

Keywords: Categorical quantum mechanics, †-compact category, completely positive maps, purification, internal trace.

1 Introduction

In [11] Selinger proposed an intriguing construction of mixed states and completely positive maps given any †-compact category representing a semantics for pure state quantum informatics in the sense of Abramsky and the author [1,2]. Conceptually speaking, in Selinger's construction an ancillary system is introduced in such a way that the distinct possible interactions between pure quantum channels and this ancillary system exactly give rise to all CPMs, and hence also all mixed states, when considering their preparation procedures as a special case of quantum channels.

Since for each †-compact category Selinger's construction provides another †-compact category, it doesn't truly provide a profound structural grasp on quantum mixedness in the usual axiomatic sense. In this paper we observe that an (admittedly quite minor) adjustment enables this construction to be recast as a true axiomatization. Moreover, this adjustment exactly imposes the preparation-state agreement axiom of [3] on the category of CPMs, that is, it explicitly requires that if two preparation procedures of pure states coincide then the resulting pure states

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should also coincide — note that while for **FdHilb** the category of finite-dimensional Hilbert spaces and linear maps **CPM**(**FdHilb**) does satisfy this requirement, for **C** an arbitrary †-compact category **CPM**(**C**) doesn't (cf. [3]+[11]).

Let's change perspective now. Given a †-symmetric monoidal category [11], passing to a †-compact category adjoins and hence axiomatizes *Bell-states* [1,2], generating at its turn all entangled states and multi-partite operations. In the same vein, in this paper we adjoin and hence axiomatize *maximally mixed states*, generating mixed states and CPMs. Moreover, the adjoints to the maximally mixed states provide and hence axiomatize the *internal traces*, which, rather than the Joyal-Street-Verity (JSV) partial traces [7] which in a †-compact category canonically arise as

$$\mathsf{Tr}^{C}_{A,B}(f : C\otimes A \to C\otimes B) :=$$
$$\lambda_B^\dagger \circ (\ulcorner 1_C \urcorner \otimes 1_B)^\dagger \circ (1_{C^*}\otimes f)\circ (\ulcorner 1_C \urcorner \otimes 1_A)\circ \lambda_A : A \to B\,,$$

play a crucial role in quantum information theory. To our knowledge, the need for an abstract notion of internal trace has so far only been indicated by Delbecque in [5], motivated by the fact that while in Selinger's construction they arise from an underlying JSV-trace in some other categories they enjoy an autonomous existence.

This same idea can also be implemented at the level of graphical calculus. While the passage from †-symmetric monoidal to †-compact introduces for each type a new primitive ingredient, e.g. 'pink triangle' in [4], which is subject to a yanking axiom, here we again introduce for each type a new primitive ingredient, which we will refer to as 'black triangle', which is again subject to some axiom. It remains to be seen how (dis)advantageous this graphical presentation is as compared to Selinger's, but it does seem to have advantages when graphically trying to conceptualize the messy zoo of all recently proposed quantum informatic quantities (e.g. [9]).

Finally, the notion of purification of mixed states and mixed channels, which plays an important role in the quantum information theory literature (e.g. [8,10]), provides a simple physical interpretation for our adaptation of Selinger's CPM-construction.

2 Denoting types and variances

For the basic definitions of †-compact categories and their interpretation as semantics for quantum mechanics we refer to the existing literature [1,2,3,11] and references therein. We will refer to †-symmetric monoidal categories as $(\otimes,\dagger)$-categories, to †-compact categories as $(\otimes,\dagger,\ulcorner 1\urcorner)$-categories, and to the categories which in addition to $(\otimes,\dagger,\ulcorner 1\urcorner)$-categories also contain maximally mixed states as $(\otimes,\dagger,\perp)$-categories (see Definition 3.1 below).

When expressing naturality we will use indices on objects to refer to the involutions $(-)^\dagger$, $(-)_*$ and $(-)^*$ which alter the variance in that variable e.g. in the case of $(\otimes,\dagger,\ulcorner 1\urcorner)$-categories

$$\mathbf{C}(\mathrm{I}, A^*\otimes B)\simeq \mathbf{C}(A,B)\simeq \mathbf{C}(A_*\otimes B^\dagger,\mathrm{I})$$

stands for commutation of

$$\begin{array}{ccccc}
\mathbf{C}(\mathrm{I}, A^* \otimes B) & \xleftarrow{\simeq} & \mathbf{C}(A,B) & \xrightarrow{\simeq} & \mathbf{C}(A^* \otimes B, \mathrm{I}) \\
\downarrow {\scriptstyle (f^* \otimes g) \circ -} & & \downarrow {\scriptstyle g \circ - \circ f} & & \downarrow {\scriptstyle - \circ (f_* \otimes g^\dagger)} \\
\mathbf{C}(\mathrm{I}, C^* \otimes D) & \xleftarrow[\simeq]{} & \mathbf{C}(C,D) & \xrightarrow[\simeq]{} & \mathbf{C}(C^* \otimes D, \mathrm{I})
\end{array}$$

and hence in ordinary compact closed categories where we have

$$\mathbf{C}(\mathrm{I}, A^* \otimes B) \simeq \mathbf{C}(A,B) \simeq \mathbf{C}(A \otimes B^*, \mathrm{I})$$

the $*$-symbol now also specifies alteration of the variance (besides merely assigning the dual object). The same convention applies to typed expressions since $f^\sharp : A^\natural \to B^\flat$ stands for $f^\sharp \in \mathbf{C}(A^\natural, B^\flat)$, and we can compress the size of the expression $f^\sharp : A^\natural \to B^\flat$ by setting $f^\sharp_{A^\natural \to B^\flat}$. Dirac notations for *states* $|\psi\rangle$ and *co-states* $\langle\psi|$ respectively arise as $\psi_{\mathrm{I}\to A}$ and $\psi^\dagger_{A^\dagger \to \mathrm{I}}$ so our notation is in fact a refinement of Dirac's by providing explicit *types* and additional data on *variances*.

When setting $C := A$, $D := C$, $f := 1_A$ and using *compositionality* [1]

$$g := \lambda_C^\dagger \circ (\ulcorner 1_{B^*} \urcorner \otimes 1_C)^\dagger \circ (1_B \otimes \ulcorner g \urcorner) \circ \rho_B : B \to C$$

in the left square of the above diagram we obtain a *natural propagation of composition* diagram

(1)
$$\begin{array}{ccc}
\mathbf{C}(A,B) \times \mathbf{C}(B,C) & \xrightarrow{- \circ -} & \mathbf{C}(A,C) \\
\simeq \downarrow & & \downarrow \simeq \\
\mathbf{C}(\mathrm{I}, A^* \otimes B) \times \mathbf{C}(\mathrm{I}, B^* \otimes C) & \xrightarrow[- \triangle -]{} & \mathbf{C}(\mathrm{I}, A^* \otimes C)
\end{array}$$

where

$$\ulcorner f \urcorner \triangle \ulcorner g \urcorner := (1_{A^*} \otimes \lambda_C)^\dagger \circ (1_{A^*} \otimes \ulcorner 1_{B^*} \urcorner \otimes 1_C)^\dagger \circ (\ulcorner f \urcorner \otimes \ulcorner g \urcorner) \circ \rho_\mathrm{I}$$

i.e. we obtain a CUT-like composition (cf. [1]).

3 Maximally mixed states, internal trace, purification

The following definition introduces *maximally mixed states* ($\perp$-states) as the *generator* of mixedness, in analogy to $\ulcorner 1 \urcorner$-states constituting the *generator* of entanglement.

Definition 3.1 A $\perp$-*structure* on a $(\otimes, \dagger)$-category $\mathbf{C}$ comprises

(i) a *maximally mixed state* $\perp_A : \mathrm{I} \to A$ for each object with $\perp_\mathrm{I} = 1_\mathrm{I}$ and $\perp_{A \otimes B} = (\perp_A \otimes \perp_B) \circ \lambda_\mathrm{I}$,

(ii) an all-objects-including sub-$(\otimes, \dagger)$-category $\mathbf{C}_\Sigma$ of *pure states* which carries a $\ulcorner 1 \urcorner$-structure,

which are such that for all $f, g \in \mathbf{C}_\Sigma$ we have

(2)
$$f \circ f^\dagger = g \circ g^\dagger \iff f \circ \perp_{\mathsf{dom}(f)} = g \circ \perp_{\mathsf{dom}(g)} .$$

In words, axiom (2) states when two mixed states $f \circ \bot_{\mathsf{dom}(f)}$ and $g \circ \bot_{\mathsf{dom}(g)}$ obtained by *acting with pure operations f and g on a maximally mixed state* $\bot_{-}$ coincide. There are two important special cases. **i.** Setting $\mathsf{dom}(f) = \mathsf{dom}(g) := \mathrm{I}$ in axiom (2) and using $\bot_{\mathrm{I}} = 1_{\mathrm{I}}$ we obtain

(3)
$$\psi \circ \psi^{\dagger} = \phi \circ \phi^{\dagger} \implies \psi = \phi$$

i.e. the *preparation-state agreement axiom* [3]. **ii.** Setting $g := 1_{\mathsf{codom}(f)}$ in axiom (2) we obtain

(4)
$$f \circ \bot_{\mathsf{dom}(f)} = \bot_{\mathsf{codom}(f)} \iff f \circ f^{\dagger} = 1_{\mathsf{codom}(f)}$$

which expresses under which pure operations the maximally mixed state remains invariant, in particular including all unitary operations. Also, from naturality of λ_A, ρ_A, $\sigma_{A,B}$, $\alpha_{A,B,C}$ and their coherence, together with $\bot_{A\otimes B} = (\bot_A \otimes \bot_B) \circ \lambda_{\mathrm{I}}$ and $\bot_{\mathrm{I}} = 1_{\mathrm{I}}$ we obtain

$$\bot_{\mathrm{I}\otimes A} = \lambda_A \circ \bot_A \quad \bot_{B\otimes A} = \sigma_{A,B} \circ \bot_{A\otimes B} \quad \bot_{(A\otimes B)\otimes C} = \alpha_{A,B,C} \circ \bot_{A\otimes(B\otimes C)} .$$

Definition 3.2 In a $(\otimes, \dagger, \bot)$-category the *partial internal trace* is the map

$$\mathsf{tr}^C_{A,B} : \mathbf{C}(A, C \otimes B) \to \mathbf{C}(A, B) :: f \mapsto \lambda^{\dagger}_B \circ (\bot^{\dagger}_C \otimes 1_B) \circ f$$

for every three objects A, B and C, and the *full internal trace* is the map

$$\mathsf{tr}^C : \mathbf{C}(\mathrm{I}, C) \to \mathbf{C}(\mathrm{I}, \mathrm{I}) :: \psi \mapsto \bot^{\dagger}_C \circ \psi$$

for every two objects A and B.

Somewhere in the middle between the partial and the full trace we encounter the cases

$$\tilde{\mathsf{tr}}^C_A : \mathbf{C}(A, C) \to \mathbf{C}(A, \mathrm{I}) :: f \mapsto \bot^{\dagger}_C \circ f$$

and

$$\mathsf{tr}^C_A : \mathbf{C}(\mathrm{I}, C \otimes A) \to \mathbf{C}(\mathrm{I}, A) :: \Psi \mapsto \lambda^{\dagger}_A \circ (\bot^{\dagger}_C \otimes 1_A) \circ \Psi .$$

Definition 3.3 In a $(\otimes, \dagger, \bot)$-category define a *purification* of an operation $f : A \to B$ to be a pure operation $g : A \to C \otimes B$ (i.e. in $\mathbf{C}_{\Sigma}$) which is such that $f = \mathsf{tr}^C_{A,B}(g)$.

An operation is *purifiable* if it admits a purification. A purifiable operation can (and usually does) admit many different purifications, even many different purifications of the same type. A special case of purifications are purifications $\Psi_{\rho} : \mathrm{I} \to C \otimes A$ of mixed states $\rho : \mathrm{I} \to A$, which play an important role in the standard quantum information theory literature.

Next we generalize the canonical JSV-traces which exist in $(\otimes, \dagger, \ulcorner 1 \urcorner)$-categories by relaxing the unit of compactness $\ulcorner 1_A \urcorner : \mathrm{I} \to A^* \otimes A$ to the name $\ulcorner f \urcorner : \mathrm{I} \to C \otimes A$ of arbitrary morphisms $f : C^* \to A$, or equivalently, by compactness, to arbitrary bipartite states $\Psi : \mathrm{I} \to C \otimes A$.

Definition 3.4 Given $\Psi : \mathrm{I} \to C \otimes A$ in a $(\otimes, \dagger, \bot)$-category the Ψ-*trace* is

$$\mathsf{Tr}(\Psi) : \mathbf{C}(A \otimes E, A \otimes E') \to \mathbf{C}(E, E') ::$$
$$f \mapsto \lambda^{\dagger}_{E'} \circ (\Psi \otimes 1_{E'})^{\dagger} \circ (1_C \otimes f) \circ (\Psi \otimes 1_E) \circ \lambda_E .$$

Denote by $\varphi_{\rho} : C^* \to A$ the pure operation which is such that $\ulcorner \varphi_{\rho} \urcorner = \Psi_{\rho}$, where Ψ_{ρ} is a purification of a mixed state ρ. Below read "Ψ_{ρ}" as "some purification of

ρ", with obvious analogue for "φ_ρ", to which we, in the vein of †-compactness, will also refer to as a purification of ρ.

The following result provides a physical interpretation for axiom 2.

Proposition 3.5 *With the assumptions of Definition 3.1 the following are equivalent:*

i. *axiom* (2),

ii. *for all purifiable* $\rho, \rho' : \mathrm{I} \to A$ *we have*

$$\varphi_\rho \circ \varphi_\rho^\dagger = \varphi_{\rho'} \circ \varphi_{\rho'}^\dagger \implies \rho = \rho' ,$$

iii. *for all purifiable* $\rho, \rho' : \mathrm{I} \to A$ *we have*

$$\mathsf{Tr}(\Psi_\rho) = \mathsf{Tr}(\Psi_{\rho'}) \implies \rho = \rho' .$$

Proof: We have $i \Leftrightarrow ii$ by the definition of φ_ρ and $ii \Leftrightarrow iii$ since by

$$\mathsf{Tr}(\Psi_\rho) = \lambda_{E'}^\dagger \circ (\ulcorner 1_A \urcorner \otimes 1_{E'})^\dagger \circ ((\varphi_\rho \circ \varphi_\rho^\dagger)^* \otimes -) \circ (\ulcorner 1_A \urcorner \otimes 1_E) \circ \lambda_E$$

and $\varphi_\rho \circ \varphi_\rho^\dagger = \mathsf{Tr}(\Psi_\rho)(\sigma_{A,A})$ it follows that $\mathsf{Tr}(\Psi_\rho)$ and $\varphi_\rho \circ \varphi_\rho^\dagger$ are in bijective correspondence. □

The last implication expresses that $\mathsf{Tr}(\Psi_\rho)$ does not depend on the particular choice of purification. This for example implies that Schumacher's [10] *entanglement fidelity* of a state ρ with respect to channel/operation $f : A \to A$, in our language defined as $\mathsf{Tr}(\Psi_\rho)(f)$, does not depend on the "particular details of the purification process".

4 Properties of purifiable operations

Denote by $\mathbf{C}^{\mathsf{purif}}$ the '$(\otimes, \dagger, \ulcorner 1 \urcorner, \bot)$-category' of all purifiable operations (see Proposition 4.1 below).

Proposition 4.1 *In a* $(\otimes, \dagger, \bot)$*-category* $\mathbf{C}$ *the* $\ulcorner 1 \urcorner$*-structure of* $\mathbf{C}_\Sigma$ *and the fact that* $\mathbf{C}_\Sigma$ *satisfies the preparation-state agreement axiom lift to* $\mathbf{C}^{\mathsf{purif}}$*, which also inherits the* $\bot$*-structure from* $\mathbf{C}$.

Proof: One easily verifies that 'purifiability' is closed under $\circ$, $\otimes$ and †, that operations in $\mathbf{C}_\Sigma$ are trivially purifiable, and in particular that $\ulcorner 1_A \urcorner$ is a purification of $\bot_A$. Hence the only non-trivial part of the proof constitutes satisfaction of preparation-state agreement. It suffices to show that for all $\rho, \rho' : \mathrm{I} \to A$ we have $\rho \otimes \rho_* = \rho' \otimes {\rho'}_* \Rightarrow \rho = \rho'$ (see [3]). For φ_ρ (resp. $\varphi_{\rho'}$) a purification for ρ (resp. ρ') we have that $\varphi_\rho \otimes (\varphi_\rho)_*$ (resp. $\varphi_{\rho'} \otimes (\varphi_{\rho'})_*$) is a purification of $\rho \otimes \rho_*$ (resp. $\rho' \otimes \rho'_*$) 'up to $\otimes$-natural isomorphisms'. We have

$$\begin{aligned}
\rho \otimes \rho_* = \rho' \otimes {\rho'}_* \\
&\Leftrightarrow (\varphi_\rho \otimes (\varphi_\rho)_*) \circ (\varphi_\rho \otimes (\varphi_\rho)_*)^\dagger = (\varphi_{\rho'} \otimes (\varphi_{\rho'})_*) \circ (\varphi_{\rho'} \otimes (\varphi_{\rho'})_*)^\dagger \\
&\Leftrightarrow (\varphi_\rho \circ \varphi_\rho^\dagger) \otimes (\varphi_\rho \circ \varphi_{\rho'}^\dagger)_* = (\varphi_\rho \circ \varphi_{\rho'}^\dagger) \otimes (\varphi_{\rho'} \circ \varphi_{\rho'}^\dagger)_* \\
&\Leftrightarrow \varphi_\rho \circ \varphi_\rho^\dagger = \varphi_\rho \circ \varphi_{\rho'}^\dagger \;\Leftrightarrow\; \rho = \rho'
\end{aligned}$$

by Proposition 3.5, bifunctoriality, preparation-state agreement for $\mathbf{C}_\Sigma$ and again Proposition 3.5 respectively, what completes this proof. □

Since $\mathsf{Tr}(\Psi_\rho)$ does not depend on the choice of purification we can denote it by Tr_ρ. More generally, due to the $\ulcorner 1 \urcorner$-structure, also for any purifiable operation $g : B \to A$ the mapping

$$\mathsf{Tr}_g : \mathbf{C}(A \otimes E, A \otimes E') \to \mathbf{C}(B \otimes E, B \otimes E') :: \\ f \mapsto (h \otimes 1_{E'})^\dagger \circ (1_C \otimes f) \circ (h \otimes 1_E)$$

where $h : B \to C \otimes A$ is any purification of $g : B \to A$ is well-defined. Recall from [3,11] that a morphism $f : A \to A$ in a $(\otimes, \dagger)$-category is *positive* if it decomposes as $f = g^\dagger \circ g$ for some morphism $g : A \to B$. Denote all purifiable states of type $\mathrm{I} \to A$ by $\mathbf{C}^{\mathsf{purif}}(\mathrm{I}, A)$ and all positive morphisms in $\mathbf{C}_\Sigma$ of type $A \to A$ by $\mathbf{C}_\Sigma^{\mathsf{pos}}(A^\dagger, A)$. We will use the notation $\simeq_\Sigma$ to denote naturality with respect to composition with pure operations.

Proposition 4.2 *Axiom (2) is equivalent to the existence of a monoidal natural bijection*

$$\mathsf{mix} : \mathbf{C}_\Sigma^{\mathsf{pos}}(A^\dagger, A) \ \simeq_\Sigma\ \mathbf{C}^{\mathsf{purif}}(\mathrm{I}, A)\,.$$

This monoidal natural bijection moreover induces commutation of

$$(5)\qquad \begin{array}{ccc} \mathbf{C}^{\mathsf{purif}}(\mathrm{I}, A^* \otimes B) \times \mathbf{C}^{\mathsf{purif}}(\mathrm{I}, B^* \otimes C) & \xrightarrow{\ -\triangle-\ } & \mathbf{C}^{\mathsf{purif}}(\mathrm{I}, A^* \otimes C) \\ \Big\downarrow{\scriptstyle \mathsf{mix}^{-1}} & & \Big\downarrow{\scriptstyle \mathsf{mix}^{-1}} \\ \mathbf{C}_\Sigma^{\mathsf{pos}}(A^* \otimes B, A^* \otimes B) \times \mathbf{C}_\Sigma^{\mathsf{pos}}(B^* \otimes C, B^* \otimes C) & \xrightarrow[\ -\diamondsuit-\ ]{} & \mathbf{C}_\Sigma^{\mathsf{pos}}(A^* \otimes C, A^* \otimes C) \end{array}$$

where $f \diamondsuit g$ is defined to be

$$(1_{A^*} \otimes \lambda_C)^\dagger \circ (1_{A^*} \otimes \ulcorner 1_{B^*} \urcorner \otimes 1_C)^\dagger \circ (f \otimes g) \circ (1_{A^*} \otimes \ulcorner 1_{B^*} \urcorner \otimes 1_C) \circ (1_{A^*} \otimes \lambda_C)\,.$$

Proof: Setting

$$\mathsf{mix} : f \circ f^\dagger \mapsto f \circ \bot_{\mathsf{dom}(f)}\,,$$

the restriction to $\mathbf{C}_\Sigma^{\mathsf{pos}}$ assures totality, the forward implication of axiom (2) assures well-definedness, the backward direction assures injectivity, restriction to $\mathbf{C}^{\mathsf{purif}}$ assures surjectivity, and monoidal naturality, i.e. commutation of

$$\begin{array}{ccc} \mathbf{C}_\Sigma^{\mathsf{pos}}(A, A) & \xrightarrow{\ \mathsf{mix}\ } & \mathbf{C}^{\mathsf{purif}}(\mathrm{I}, A) \\ {\scriptstyle g \circ - \circ g^\dagger}\Big\downarrow & & \Big\downarrow{\scriptstyle g \circ -} \\ \mathbf{C}_\Sigma^{\mathsf{pos}}(B, B) & \xrightarrow[\ \mathsf{mix}\ ]{} & \mathbf{C}^{\mathsf{purif}}(\mathrm{I}, B) \end{array}$$

where g is pure together with 'good' behavior of mix w.r.t. $\otimes$, follow straightforwardly — note in particular that the action $g \circ - \circ g^\dagger : \mathbf{C}_\Sigma^{\mathsf{pos}}(A^\dagger, A) \to \mathbf{C}^{\mathsf{pos}}(B^\dagger, B)$ indeed preserves positivity of morphisms. When setting $\bot_A := \mathsf{mix}(1_A)$ the converse is also straightforward. For $f : D \to A \otimes C$ pure and $h : B \to C \otimes A$ a co-purification of $g : B \to A$ we have

$$\mathsf{mix}(\underbrace{h \circ (1_C \otimes f \circ f^\dagger) \circ h^\dagger}_{\mathsf{Tr}_g(f \circ f^\dagger)}) = h \circ (1_C \otimes f) \circ \bot_{C \otimes D} = \underbrace{h \circ (\bot_C \otimes 1_D) \circ \lambda_D}_{g} \circ \mathsf{mix}(f \circ f^\dagger)$$

so we also have commutation of the more general diagram

$$\begin{array}{ccc} \mathbf{C}_\Sigma^{\mathsf{pos}}(A, A) & \xrightarrow{\mathsf{mix}} & \mathbf{C}^{\mathsf{purif}}(\mathrm{I}, A) \\ {\scriptstyle \mathsf{Tr}_g}\downarrow & & \downarrow{\scriptstyle g \circ -} \\ \mathbf{C}_\Sigma^{\mathsf{pos}}(B, B) & \xrightarrow[\mathsf{mix}]{} & \mathbf{C}^{\mathsf{purif}}(\mathrm{I}, B) \end{array} \tag{6}$$

where g now only has to be purifiable. Diagram (5) now also easily follows. □

From diagram (6) in the above proof it follows that axiom (2) in Definition 3.1 can in fact be extended from pure operations to all purifiable operations.

Corollary 4.3 *In a $(\otimes, \dagger, \bot)$-category for all $f, g \in \mathbf{C}^{\mathsf{purif}}$ we have*

$$\mathsf{Tr}_f = \mathsf{Tr}_g \iff f \circ \bot_{\mathsf{dom}(f)} = g \circ \bot_{\mathsf{dom}(g)} . \tag{7}$$

5 Recovering Selinger's CPM-construction

Denote by $\mathbf{C}^{\mathsf{pos}}$ the graph with the same objects as $\mathbf{C}$ but morphisms restricted to the positive ones.[3] We will now present Selinger's CPM-construction of [11], slightly modified such that it fits better the needs of this paper. Given a $(\otimes, \dagger, \ulcorner 1 \urcorner)$-category $\mathbf{C}$ define a new category $\mathbf{CPM}(\mathbf{C})$ which has the same objects as $\mathbf{C}$, but which has as morphisms

$$\mathbf{CPM}(\mathbf{C})(A, B) := \mathbf{C}^{\mathsf{pos}}(A^* \otimes B, A^* \otimes B)$$

with $\diamondsuit$ as composition and hence which has $\ulcorner 1_A \urcorner \circ (\ulcorner 1_A \urcorner)^\dagger$ as identities. Selinger went on showing that $\mathbf{CPM}(\mathbf{C})$ is again a $(\otimes, \dagger, \ulcorner 1 \urcorner)$-category and in particular that $\mathbf{CPM}(\mathbf{FdHilb})$ is the category which has completely positive maps as morphisms and (not necessarily normalized) density matrices as its *elements* i.e. morphisms with as type $\mathbb{C} \to \mathcal{H}$. Note here that if $f \in \mathbf{CPM}(\mathbf{C})(A, B) = \mathbf{C}^{\mathsf{pos}}(A^* \otimes B, A^* \otimes B)$ then by positivity $f = g^\dagger \circ g$, and each choice for such a $g^\dagger : C \to A^* \otimes B$ yields in fact a purification for the operation f in the sense of Section 3.

Theorem 5.1 *If $\mathbf{C}$ carries a $\bot$-structure then $\mathbf{CPM}(\mathbf{C}_\Sigma) \simeq \mathbf{C}^{\mathsf{purif}}$.*

Proof: By Proposition 4.2 we have

$$\mathbf{C}^{\mathsf{purif}}(A, B) \simeq \mathbf{C}^{\mathsf{purif}}(\mathrm{I}, A^* \otimes B) \simeq \mathbf{C}_\Sigma^{\mathsf{pos}}(A^* \otimes B, A^* \otimes B) \stackrel{\text{Def.}}{=} \mathbf{CPM}(\mathbf{C}_\Sigma)(A, B)$$

and diagrams (1) and (5) guarantee that also composition carries over. □

[3] Note that above we implicitly made the convention $\mathbf{C}_\Sigma^{\mathsf{pos}} := (\mathbf{C}_\Sigma)^{\mathsf{pos}}$.

Selinger also introduced the canonical identity-on-objects mapping

$$F_{\mathbf{CPM}}:\mathbf{C}\to\mathbf{CPM}(\mathbf{C})::f\mapsto\ulcorner f\urcorner\circ(\ulcorner f\urcorner)^\dagger$$

which due to the variances (cf. composition in $\mathbf{CPM}(\mathbf{C})$ is $\diamondsuit$)

$$\mathbf{C}(A,B)\xrightarrow{F_{\mathbf{CPM}}}\mathbf{C}(A_*\otimes B^\dagger,A^*\otimes B)=:\mathbf{CPM}(\mathbf{C})(A,B) \tag{8}$$

provides a functorial passage from $\mathbf{C}$ to $\mathbf{CPM}(\mathbf{C})$, and the intended interpretation of the range of this functor are pure operations/states. In general $F_{\mathbf{CPM}}$ is not faithful and this is due to the fact that in general $\mathbf{C}$ does not satisfy preparation-state agreement. [4]

Lemma 5.2 *For a $(\otimes,\dagger,\ulcorner 1\urcorner)$-category $\mathbf{C}$ the following are equivalent:*

1. $\mathbf{C}^{\mathsf{pos}}$ *satisfies the preparation-state agreement axiom*;
2. $\mathbf{C}^{\mathsf{pos}}\simeq F_{\mathbf{CPM}}[\mathbf{C}^{\mathsf{pos}}]$;
3. $\mathbf{CPM}(\mathbf{C})$ *satisfies the preparation-state agreement axiom*;
4. $\mathbf{CPM}(\mathbf{C})\simeq\mathbf{CPM}(F_{\mathbf{CPM}}[\mathbf{C}])$;
5. $\mathbf{CPM}(\mathbf{C})\simeq\mathbf{CPM}(\mathbf{C}')$ *for some* $\mathbf{C}'$ *which satisfies preparation-state agreement*;

where all isomorphisms are assumed to be canonical ones.

Proof: Equivalences $\mathbf{1}\Leftrightarrow\mathbf{2}$ and $\mathbf{3}\Leftrightarrow\mathbf{4}$ follow by the fact that the preparation-state agreement axiom can be stated as $f=g\Leftrightarrow F_{\mathbf{CPM}}(f)=F_{\mathbf{CPM}}(g)$, and $\mathbf{1}\Leftrightarrow\mathbf{3}$ follows straightforwardly by the definition of $\mathbf{CPM}(\mathbf{C})$. $\mathbf{3},\mathbf{4}\Rightarrow\mathbf{5}$: if $\mathbf{CPM}(\mathbf{C})$ satisfies the preparation-state agreement axiom then so does $F_{\mathbf{CPM}}[\mathbf{C}]$, hence $\mathbf{5}$ follows by $\mathbf{4}$ for $\mathbf{C}':=F_{\mathbf{CPM}}[\mathbf{C}]$. $\mathbf{5}\Rightarrow\mathbf{3}$: if $\mathbf{C}'$ satisfies the preparation-state agreement axiom then so does $\mathbf{CPM}(\mathbf{C}')$ and hence so does $\mathbf{CPM}(\mathbf{C})$. □

The equivalent conditions **1–5** in Lemma 5.2 do not require $\mathbf{C}$ itself to satisfy the preparation-state agreement axiom i.e., equivalently, $\mathbf{C}\simeq F_{\mathbf{CPM}}[\mathbf{C}]$. A counter example is **FdHilb**. But they are slightly stronger than only requiring that $F_{\mathbf{CPM}}[\mathbf{C}]$ satisfies the preparation-state agreement axiom i.e., equivalently, $F_{\mathbf{CPM}}[\mathbf{C}]\simeq F_{\mathbf{CPM}}[F_{\mathbf{CPM}}[\mathbf{C}]]$.

Theorem 5.3 *If $\mathbf{C}$ is a $(\otimes,\dagger,\ulcorner 1\urcorner)$-category then*

$$\bot_A:=F_{\mathbf{CPM}}(1_A)\qquad\text{and}\qquad\mathbf{CPM}(\mathbf{C})_\Sigma:=F_{\mathbf{CPM}}[\mathbf{C}]$$

define a $\bot$-structure on $\mathbf{CPM}(\mathbf{C})$ iff the equivalent conditions **1–5** *in Lemma 5.2 hold.*

Proof: Since $\mathbf{CPM}(\mathbf{C})_\Sigma(A,B)=F_{\mathbf{CPM}}[\mathbf{C}(A,B)]$ and the fact that positivity is a compositionally defined property with $F_{\mathbf{CPM}}$ being functorial we have

$$\mathbf{CPM}(\mathbf{C})_\Sigma^{\mathsf{pos}}(A^\dagger,A)=F_{\mathbf{CPM}}[\mathbf{C}^{\mathsf{pos}}(A^\dagger,A)]\,.$$

Hence, since we also have that

$$\mathbf{CPM}(\mathbf{C})^{\mathsf{purif}}(\mathrm{I},A)\stackrel{\text{Def.}}{=}\mathbf{C}^{\mathsf{pos}}(\mathrm{I}_*\otimes A^\dagger,\mathrm{I}^*\otimes A)\simeq\mathbf{C}^{\mathsf{pos}}(A^\dagger,A)$$

[4] In [3] the preparation-state agreement axiom was derived as a fixed point with respect to $F_{\mathbf{CPM}}$, which was introduced as a construction which 'eliminates global phases', independent of the Selinger's **CPM**-construction.

condition **2** in Lemma 5.2 (i.e. the restriction of $F_{\mathbf{CPM}}$ to $\mathbf{C}^{\mathsf{pos}}$ is faithful) suffices in the light of Proposition 4.2 to establish a $\perp$-structure on $\mathbf{CPM}(\mathbf{C})$. □

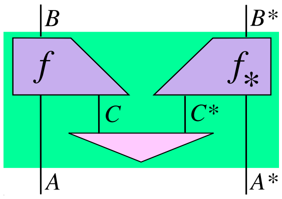
Hence we can indeed conclude that:

$$\perp\text{-structure} \quad\equiv\quad \text{CPM-construction} + \text{preparation-state agreement}$$

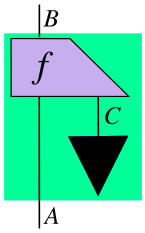
That is, more precisely, carrying a $\perp$-structure coincides with the subcategory of purifiable operations being isomorphic to a category $\mathbf{CPM}(\mathbf{C})$ which is the result of applying Selinger's CPM-construction to a category $\mathbf{C}$ which satisfies the preparation-state agreement axiom (cf. **5** in Lemma 5.2), and this satisfaction of the preparation-state agreement axiom of that underlying category in turns coincides with the subcategory of purifiable operations, or equivalently, $\mathbf{CPM}(\mathbf{C})$ itself satisfying the preparation-state agreement axiom (cf. **3** in Lemma 5.2).

6 Introducing black triangle, and outlook

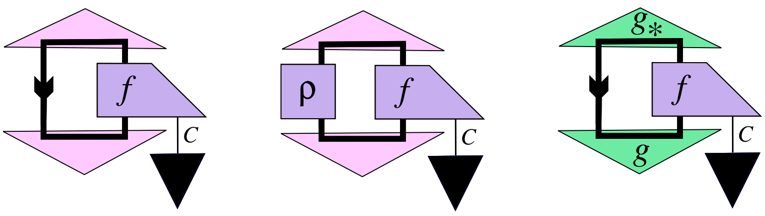
Graphically (cf. [4]), Selinger's CPM-construction, of which we now consider the covariant version of [11] (and not the version considered above), boils down to 'restricting' to operations of the shape:



where $f : A \otimes C \to B$ is a (co)purification of the operation of type $A \to B$ under consideration. This pictures carries some sort of redundancy in that they both involve f and a copy of it subjected to $(-)_*$. We can reduce this notation by introducing a new primitive notion, referred to above as maximally mixed states, and depicted as a black triangle:



which is subject to the graphical counterpart to axiom (2). In this representation quantitative notions such as Reimpell and Werner's *channel fidelity*, Schumacher's *entanglement fidelity* and Devetak's *entanglement generating capacity* (see [9] and references therein) emerge naturally as:



We intend to systematically analyse these important quantitative notions of quantum information theory in this qualitative manner, and cast them within a uniform theory. We expect that new canonical and unifying notions will emerge. This work is still in progress, and hence is not fully represented here, but we do expect a compositional theory on quantum informatic resources to emerge, which substantially extends the recent proposals by Devetak, Harrow and Winter in [6].

We also observe that the $\perp$-structures introduced in this paper have, conceptually speaking, a domain-theoretic flavour: they involve an analogue bottom-elements, namely the maximally mixed states, and an analogue to maximal elements, namely the pure operations, while the other relevant operations are obtained by acting on the bottom-elements (axiom (2)), a fact which is domain theory is key to the fixed-point theorem. Also here we see an opportunity for substantial elaboration, in the same vein as categorical logic extends algebraic logic.

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